# A UNIFIED APPROACH TO THE MASS BALANCING OF ROTATING FLEXIBLE SHAFTS

By

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A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF
THE UNIVERSITY OF FLORIDA
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1980

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by

Mark S. Darlow

With love to my wife, Myra, and our son, Adam

## ACKNOWLEDGMENTS

The author expresses his appreciation to Dr. George N. Sandor for his guidance and supervision in the preparation of this dissertation.

Appreciation is given to the following committee members for their suggestions and supervision:

Professor Robert B. Gaither

Professor George Piotrowski

Professor Edward K. Walsh

Professor Donald W. Hearn

The author extends appreciation to Mechanical Technology Incorporated, Latham, New York, for their considerable support, both financial and technical. Particular appreciation is given to Dr. Anthony J. Smalley, Manager, Machinery Dynamics Section, Mechanical Technology Incorporated, for his continuing interest and support, and for serving as an unofficial committee member.

Finally the author extends his deepest appreciation to his family for their support and encouragement.

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## LIST OF SYMBOLS

### Roman Symbology

$A$: Cross-sectional area of rotor

$\hat{\underset{\sim}{A}}$: Complex influence coefficient matrix

$a$: Pure radial mass eccentricity of rotor

$\hat{a}(z)$: Total rotor complex-valued mass eccentricity

$\hat{a}_i$: Complex modal mass eccentricity for $i^{th}$ mode

$a_u(z)$, $a_v(z)$: Components of $\hat{a}(z)$ in rotating coordinate system

$a_{iu}$, $a_{iv}$: Components of $\hat{a}_i$ in rotating coordinate system

$b$: Viscous damping coefficient

$\hat{\underset{\sim}{b}}$: Vector of complex uncorrected rotor vibration data

$\hat{b}_i$: $i^{th}$ complex element of $\hat{\underset{\sim}{b}}$

$E$: Modulus of elasticity

$\underset{\sim}{E}_k$: Influence coefficient iterative weighting matrix for $k^{th}$ iteration

$e$: Transcendental number equal to 2.71828182 . . .

$\hat{\underset{\sim}{e}}_i$: $i^{th}$ complex-valued basis vector

$\underset{\sim}{F}$: Influence coefficient general weighting matrix

$f_i$: $i^{th}$ diagonal element of $\underset{\sim}{F}$

$I$: Cross-sectional area moment of inertia; also used to represent $I_p - I_t$

$I_p$: Polar (gyroscopic) moment of inertia

$I_t$: Transverse (rotatory) moment of inertia

$i$: Square root of negative one

$k$: Linear stiffness coefficient (spring rate)

$\ell$: Generally used to indicate rotor length

$m$: Mass

$m_i$: $i^{th}$ added mass for balancing

$O$: Origin of stationary and rotating coordinate systems

$Q$: Modal normalizing factor

$\hat{R}$: Complex value representing rotor vibration in a rotating coordinate system

$r$: Magnitude of $\hat{R}$

$\hat{r}_i$: Complex value representing radius and angular location of $m_i$

$S_f$: Significance factor used for balancing plane optimization

$t$: Time

$U$: Abscissa in rotating coordinate system

$u$: Component of vibration vector along U axis

$\hat{u}$: Complex-valued rotor mass imbalance

$u_c$, $u_s$: Components of $\hat{u}$ in rotating coordinate system

$V$: Ordinate in rotating coordinate system

$v$: Component of vibration vector along V axis

$\hat{\underset{\sim}{w}}$: Vector of complex-valued trial or correction masses

$\hat{w}_i$: $i^{th}$ component of $\hat{\underset{\sim}{w}}$

$X$: Abscissa in stationary coordinate system

$x$: Component of vibration vector along X axis

$\hat{x}$: Complex-valued rotor vibration amplitude

$x_c$, $x_s$: Components of $\hat{x}$ in rotating coordinate system

$\hat{\underset{\sim}{x}}$: Vector of complex vibration values

$\hat{\underset{\sim}{x}}_w$: Vibration vector due to mass set $\hat{\underset{\sim}{w}}$

Y: Ordinate in stationary coordinate system

y: Component of vibration vector along Y axis

Z: Coordinate lying along rotor axis

z: Distance from origin along rotor axis

## Greek Symbology

$\alpha_{ij}$: $ij^{th}$ element of influence coefficient matrix

$\beta$: Lagging phase angle in Jeffcott analysis

$\beta_\theta$: Lagging phase angle in conical Jeffcott analysis

$\varepsilon_{ik}$: $i^{th}$ influence coefficient weighting factor for $k^{th}$ iteration

$\zeta$: Damping ratio for Jeffcott analysis

$\zeta_i$: Damping ratio for $i^{th}$ mode in modal analysis

$\zeta_\theta$: Damping ratio for conical Jeffcott analysis

$\hat{\eta}(z)$: Total rotor whirl deformation

$\hat{\eta}_i$: Rotor whirl deformation for $i^{th}$ mode

$\eta_u(z)$, $\eta_v(z)$: Components of $\hat{\eta}(z)$ in rotating coordinate system

$\eta_{iu}$, $\eta_{iv}$: Components of $\hat{\eta}_i$ in rotating coordinate system

$\theta$: Angular dynamic displacement of rotor

$\theta_a$: Pure angular mass eccentricity of rotor

$\lambda_i$: $i^{th}$ rotor system eigenvalue

$\Pi$: Indicates mathematical multiplication series

$\pi$: Standard constant 3.14159265 . . .

$\rho$: Mass density

$\sum$: Indicates mathematical summation series

$\phi$: Leading phase angle

$\phi_i(z)$: $i^{th}$ rotor system eigenvector (mode shape)

$\omega$: Rotational frequency of rotor

$\omega_{cr}$: Rotor critical speed for Jeffcott analysis

$\omega_i$: Undamped critical speed for $i^{th}$ mode in modal analysis

$\omega_n$: Undamped natural frequency

$\omega_{n\theta}$: Undamped angular natural frequency

## Explanation of Notation

$\dot{f}$: Indicates differentiation with respect to time

$\hat{f}$: Indicates complex quantity (often dropped for convenience where no ambiguity exists)

$\underset{\sim}{f}$: Indicates (real or complex) vector quantity

$\underset{\sim}{F}$: Indicates (real or complex) matrix quantity

$f_c$, $f_s$: Indicates components in rotating coordinate system

$f_u$, $f_v$: Indicates components in rotating coordinate system

$f_x$, $f_y$: Indicates components in stationary coordinate system

Abstract of Dissertation Presented to the Graduate Council
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

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June 1980

Chairman: George N. Sandor
Major Department: Mechanical Engineering

Flexible rotor balancing is essential for the safe operation of many types of rotating machines. The existing flexible rotor balancing procedures may be divided into the classifications of modal and influence coefficient balancing, none of which have gained broad acceptance due to the inherent disadvantages of each method.

In this study, a new flexible rotor balancing procedure which incorporates the advantages (and objectives) and eliminates the disadvantages of modal and influence coefficient balancing has been developed, implemented and verified. The analytical background for, and a discussion of the literature of, rotor balancing is presented to provide a basis for this study.

An original balancing plane optimization procedure for influence coefficient balancing has also been developed in this study. This procedure was designed to eliminate the common problem of redundant balancing planes in influence coefficient balancing.

A new balancing procedure is analytically developed in this study and is referred to as the Unified Balancing Approach. A detailed procedure is outlined for its implementation and test results are reported which verify the effectiveness of the Unified Balancing Approach. These results demonstrate its superiority over modal and influence coefficient balancing, in terms of both efficiency and effectiveness. The Unified Balancing Approach has been specifically designed to take into account practical rotor balancing considerations. This is of particular importance since the ultimate test of any rotor balancing method must involve practical applications.

Combined with modern microcomputer hardware, the Unified Balancing Approach forms an operation-oriented balancing tool which is ideally suited to fill the current gap in flexible rotor balancing, particularly for use in the field. The design of such a portable balancing tool is described in detail in this study.

# CHAPTER 1
# INTRODUCTION

The mass balancing of flexible rotors is an integral part of the study and practice of rotordynamics. As is implied by the name, rotordynamics is concerned with the dynamics of rotating machinery. In this context, dynamics refers to harmonic motion, or vibration. Any machine with rotating components is considered to be a rotating machine. The great majority of commercial machinery falls into this category. Rotating machines range in size from small gyroscopes weighing only a few ounces to large rock tumblers weighing several tons. A few other examples of rotating machines are household electric motors, internal combustion engines, gas turbine engines, steam turbines, electric generators, industrial compressors and power transmission systems, to name just a few.

The part of a rotating machine that rotates in normal operation is generally referred to as the rotor. The part that does not rotate is referred to as the stator. The rotor is supported by bearings which are attached to the stator. Rotating machines are generally divided into two categories, according to whether the rotor is rigid or flexible in operation. A rotor is flexible if it deforms when the machine is operating at any speed up to its maximum operating speed (including designed overspeed conditions). Otherwise, the rotor is rigid. Of principal interest here are machines with flexible rotors, as is discussed below.

The general purpose of the study of rotordynamics is to increase understanding of rotor vibration phenomena and thus provide a means for controlling or eliminating these vibrations. As the efficiency of rotating machinery has been increased through reduced weight and increased speed (which implies increased rotor flexibility) the control of rotor vibrations has become essential. Noise reduction was at one time the primary motivation for the control of rotor vibrations. However, for current machinery designs, control of rotor vibration is necessary to ensure machinery survival and operator safety. In the case of commercial aircraft, the safety of hundreds of passengers (per flight) is the primary consideration. Thus, the study and practice of rotordynamics has taken on an increasingly important role in recent years.

Rotor vibration is generally classified according to the direction and frequency of the vibratory motion. The three possible directions of motion, illustrated in Figure 1-1, are lateral, axial, and torsional. Lateral motion involves a displacement from its central position, flexural deformation and/or rotation of the rotor's axis of rotation. In this context, rotation is about an axis intersecting and normal to the axis of rotation. Axial motion occurs parallel to the rotor's axis of rotation. Torsional motion involves a rotation of the rotor's transverse sections relative to one another about its axis of rotation. Vibrations which occur at the frequency of rotation of the rotor are referred to as synchronous vibrations. Those which occur at other frequencies are known as nonsynchronous vibrations. In geared systems, where two or more rotors are operating at different speeds,

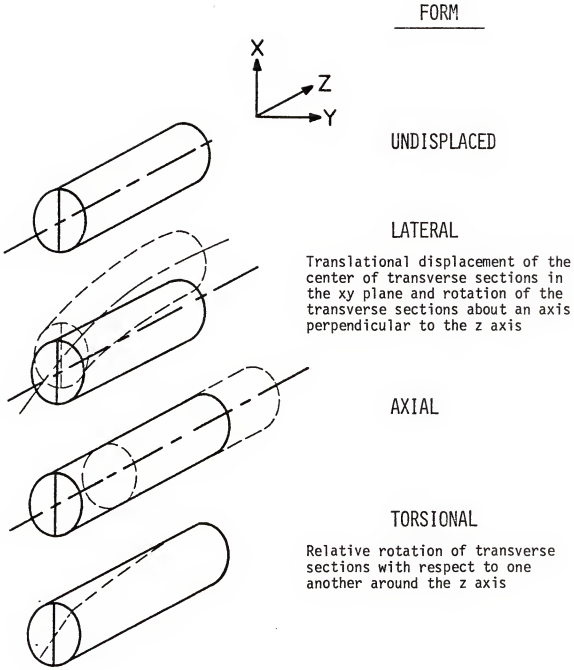


Figure 1-1 Forms of rotor vibration

a different synchronous frequency is associated with each rotor. Synchronous lateral vibration is a frequent cause of machine failure and probably the most common source of machine noise and vibration. Rotating mass unbalance results only in synchronous lateral vibration and is its most common cause. Thus, other forms of rotor vibration are of little interest in a study of rotor balancing, such as is presented here. Consequently, further reference to rotor vibration generally implies synchronous lateral vibration.

The presence of rotor mass unbalance (hereafter referred to simply as unbalance) is due to the eccentricity of the mass-centroidal axis of the rotor relative to its axis of rotation. In practice, this unbalance is generally a result of unavoidable imperfections in rotor manufacture and assembly. The term "rotor balancing" covers a broad range of procedures aimed at reducing rotor unbalance. The ultimate purpose of all rotor balancing procedures is to produce a smooth-running machine. This is achieved in practice not by removing all unbalance, but rather by applying compensating unbalances (or by removing mass by machining) which result in a smooth-running machine. Since rotor unbalance is generally distributed along the length of the rotor and compensating unbalances are applied at discrete axial locations, the complete elimination of unbalance is usually impossible, as well as unnecessary.

Rotor balancing procedures are classified according to the type of rotor for which they are designed: rigid or flexible. When balancing rigid rotors, the shape of the centroidal axis does not change as a function of speed. For flexible rotors, the shape of the

centroidal axis, and thus the unbalance distribution does change with speed. Thus, in general, rigid rotor balancing procedures and tools are not applicable to the balancing of flexible rotors. The procedures for balancing rigid rotors are well established and accepted. However, the same is not true for the current flexible rotor balancing procedures. These procedures can be classified into two groups, modal balancing* and influence coefficient balancing.** These two general balancing methods have tended to compete in the past each with its own inherent advantages and disadvantages (enumerated in a later chapter). In both cases, the disadvantages have been significant enough to prevent widespread acceptance of the balancing procedure. Consequently, in practice most flexible rotor balancing is currently done by one of several trial-and-error procedures, which are extremely inefficient.

A unified approach to the balancing of flexible rotors is proposed herein to bridge the gap between modal and influence coefficient balancing. This new balancing method (hereafter referred to as the Unified Balancing Approach, or UBA) is designed to combine the advantages of modal and influence coefficient balancing, while eliminating the disadvantages of the two current procedures. At the same time, the Unified Balancing Approach provides a straightforward, practically oriented approach to flexible rotor balancing which is well

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*Modal balancing assumes orthogonal modes of vibrational deformation and strives to compensate for them.

**Influence coefficient balancing determines the effect of a unit unbalance mass at various stations along the z axis and selects a set of discrete correction masses that will minimize vibrational amplitudes along the z axis.

suited to both production (during manufacture) and field ("in situ") balancing applications. While the objectives of the Unified Balancing Approach are similar to those of its predecessors, the actual procedure is an entirely new development.

Thus, this original contribution provides a rotor balancing tool which has been shown in actual practice to be superior to existing tools for the balancing of flexible rotors. This will result in the improvement in the state of the art of rotor balancing in both an academic and a practical sense. In particular, the applications to production and field balancing will bring about improvements in both the efficiency and the effectiveness of rotor balancing during manufacture as well as in maintenance operations.

The hardware requirements have been specified and the basic design laid out for a field-portable microprocessor-controlled UBA balancing system. This system is compact, lightweight, fully self-contained and easy to use, all of which are necessary attributes of a field-balancing system. In this system design, the software for implementation of the Unified Balancing Approach is stored in firmware (permanent read-only memory). Also, the operator's instructions would be written in such a way as to permit use by moderately unskilled technicians who do not have an extensive understanding of balancing theory.

The development and verification of the Unified Balancing Approach has consisted of five steps:

1. Developed the basic analytical approach required to take advantage of modal response while retaining

the empirical nature of influence coefficient balancing. This has been accomplished through the use of an influence coefficient manipulation procedure.

2. Incorporated the principal features of influence coefficient balancing (notably, evaluation of measurement error and least squares minimization) into the Unified Balancing Approach. This has been done by suitable modification and extension of the basic analytical procedures.
3. Developed a logical procedure for the Unified Balancing Approach which may be efficiently applied to a variety of applications including laboratory tests, production balancing and field balancing. This required careful evaluation of the requirements for these applications and how they relate to a general balancing procedure.
4. Developed the software required for applying the Unified Balancing Approach. This has been done based on the use of an on-line computer, resulting in the most efficient balancing system.
5. Experimentally verified the effectiveness of the Unified Balancing Approach. This was done through the use of a balancing test rig including comparative influence coefficient and modal balancing tests.

In its final form, the procedure for applying the Unified Balancing Approach resembles influence coefficient balancing in its empirical nature. However, the actual balancing calculation and correction procedures are entirely different. Generally, the Unified Balancing Approach is used to balance one mode at a time (though this is not a theoretical limitation). Through the use of trial mass data and previously determined influence coefficients, a set of trial masses is constructed which excites the mode being balanced while resulting in negligible response of the previously balanced modes. This trial mass set is then used in a manner similar to single plane influence coefficient balancing, to determine the complex multiple of that trial mass set which minimizes the response of the mode being balanced. This correction mass set does not excite the previous balanced modes. Enhancements to this basic procedure have been implemented which increase the flexibility of the Unified Balancing Approach and reduce the required number of trial mass runs (and, thus, speed the balancing process).

One of the principal disadvantages of influence coefficient balancing involves the inadvertent use of redundant balancing planes. This can result in ill-conditioned (nearly singular) influence coefficient matrices which can lead to the calculation of unreasonably large and essentially self-canceling correction masses. A balancing plane optimization procedure for influence coefficient balancing is also introduced. This original development is designed to eliminate this problem of redundant balancing planes in influence coefficient balancing.

The following topics are discussed in subsequent chapters: (1) a basic review of vibration and rotordynamic theory, (2) a review of the literature on rotor balancing (particularly flexible rotor balancing), (3) a detailed review and discussion of rigid rotor balancing and the current flexible rotor balancing procedures, (4) a proposed enhancement of the influence coefficient balancing method (balancing plane optimization procedure) designed to eliminate the possibility of using redundant balancing planes, (5) the theoretical development and practical procedures for the Unified Balancing Approach, and (6) experimental verification of the Unified Balancing Approach. A glossary of terms is also included along with appendices presenting abstracts of literature directly related to the UBA and detailed literature reviews of other rotor balancing literature. An extensive list of references follows.

# CHAPTER 2
# THEORETICAL BACKGROUND

In recent years, the trend in the design of rotating machinery has been toward reduced weight and increased operating speeds, with the objective of increasing operating efficiency and thus reducing cost. However, these more efficient designs result in increased rotor flexibility and are, in general, more susceptible to a variety of undesirable rotordynamic phenomena. In particular, increased rotor flexibility substantially complicates rotor balancing requirements. Control of rotor vibration is necessary to maintain reasonable noise levels and ensure operator (and consumer) safety and machinery survival. In general, the cost of operation of rotating machinery is a direct function of the success with which rotor vibrations are controlled.

Rotor vibration is generally classed into two groups. The first group is synchronous vibration which is usually controlled through rotor balancing. The second group is nonsynchronous vibration, which is often unstable and is usually controlled through the addition of external damping. The former of these groups is of interest here.

To fully appreciate the rotor balancing problem, an understanding of basic vibration theory is necessary. A brief overview of unidirectional vibration theory is presented in this chapter for this purpose. Many references are available which provide a much more

complete presentation of vibration theory, among them references [1, 2, 3, and 4].

An introduction to the theory of rotor-bearing vibrations is also presented in this chapter, with emphasis on synchronous vibration. A relatively few references are available which provide a complete treatment of rotor-bearing vibration. The most prominent of these are references [5, 6, 7, and 8].

## 2.1 Basic Vibration Theory

A vibratory system may be considered to be composed of a number of individual elements, each with the capability to store potential and kinetic energy and to dissipate energy. The vibration phenomenon is a manifestation of an alternating transfer of energy between kinetic and potential. For simplicity, system representations are generally used whose elements each have only one of the three capabilities mentioned above. These representations are referred to as lumped parameter models. The number of these elements required to accurately represent a real vibratory system depends on a number of factors including the geometry and loading of the system and the anticipated excitation frequencies.

For the purposes of this presentation, these lumped parameter model elements are assumed to behave linearly. This is, in general, a reasonable assumption for small vibratory motions, although it may not be as accurate for large motions. In the rotor balancing problem, the small motion assumption is rarely violated. In any case, most balancing procedures can tolerate modest system nonlinearities, and

the Unified Balancing Approach, through the use of repeated balancing, can efficienty handle rotor-bearing systems with substantial non-linearity.

The three lumped parameter model elements are the spring for storing potential energy, the mass for storing kinetic energy and the damper for dissipating energy. A linear spring exerts a force proportional to its change in length and in such a direction as to attempt to restore the spring to its original length. This type of force is commonly referred to as a restoring force. The proportionality constant between the deflection of the spring and the restoring force is alternately known as spring stiffness, spring constant or spring rate, and is represented by the letter k. This ideal (lumped parameter) spring is massless so that the forces at the ends of the spring are equal in magnitude and opposite in direction.

For a mass, which stores kinetic energy, the force is proportional to acceleration. The proportionality constant here is mass (as distinguished from weight) and is represented by the letter m. A lumped mass element, used for modeling purposes, is assumed to be rigid.

Of several possible representations for an energy dissipation element, the viscous damper is most commonly used. A discussion of other damper models is beyond the scope of this study. The restoring force for a viscous damper is proportional to the relative velocity between the attachment points (ends) of the damper. The proportionality constant, referred to as the viscous damping constant or coefficient, is represented by the letter b. The ideal damper, like

the ideal spring, is massless so that the forces at the ends are equal in magnitude and opposite in direction.

The simplest possible vibratory system representation is a single-degree-of-freedom model for free vibration with no damping, as illustrated schematically in Figure 2-1. The term "free vibration" indicates that the system is provided with a set of initial conditions and then allowed to vibrate freely, with no external influences. For the model in Figure 2-1, a mass is attached to a rigid support by a linear spring. The motion of the mass is uni-directional with displacement x, where x is a function of time. The differential governing equation for this system may be obtained through the use of Newton's second law of motion, which may be stated as, "the rate of change of momentum is equal to the vector sum of forces exerted on the body and takes place in the direction in which the force (vector sum) acts." If the mass is constant, the rate of change of momentum is equal to the mass times its acceleration.

Since the system considered here is unidirectional, this law reduces to

$$m\ddot{x} = \Sigma \text{ (forces in the X-direction)} = -kx \tag{2-1}$$

where the dots over the x represent derivatives with respect to time and the negative sign indicates that the spring force is a restoring force. The form of the restoring force indicates that the static equilibrium position is assumed to be at x equals zero.

Rearrangement of Equation (2-1) results in the more familiar form of the governing equation given by

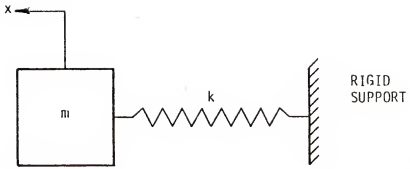



Figure 2-1 Single-degree-of-freedom free vibrational model with no damping

$$m\ddot{x} + kx = 0 \tag{2-2}$$

The solution to this differential equation is given by

$$x = A \sin \omega_n t + B \cos \omega_n t \tag{2-3}$$

where the constants A and B are determined from the initial conditions of the system and $\omega_n$, known as the natural frequency or resonance, is given by

$$\omega_n = \sqrt{k/m} \tag{2-4}$$

The units of $\omega_n$ are radians per second.

Equation (2-3) may be rewritten in the form

$$x = C \sin (\omega_n t + \theta) \tag{2-5}$$

where C is maximum amplitude of vibration (single amplitude) given by

$$C = \sqrt{A^2 + B^2} \tag{2-6}$$

and $\theta$ is the phase angle of vibration given by

$$\theta = \arctan (B/A) \tag{2-7}$$

The phase angle is an indication of the point in the vibration cycle at which the value of x becomes equal to zero. The term "phase angle," when used in different contexts, can take on different meanings. This form of vibration is referred to as simple harmonic motion due to the harmonic (cyclic) form of Equations (2-3) and (2-5).

The vibratory system of Figure 2-1 is made slightly more complicated by the addition of a viscous damping element as shown in Figure 2-2. Again considering only free vibration, the differential governing equation can be derived from Newton's second law to give

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{2-8}$$

where b is the viscous damping coefficient and the other parameters have been defined above. The form of the solution to this differential equation depends on the value of b relative to the values of m and k (assuming all are positive). The criteria for the selection of the appropriate solution is whether the damping coefficient, b, is less than, equal to or greater than the critical damping value, $b_c$, which is defined by

$$b_c = 2\sqrt{km} = 2m\omega_n \tag{2-9}$$

where $\omega_n$ is defined as before to be $\sqrt{k/m}$. The ratio of the damping coefficient to the critical damping is referred to as the damping ratio, $\zeta$.

For the case in which the damping is less than critical, or underdamped ($\zeta < 1$), the solution to equation (2-8) is of the form

$$x = e^{-bt/2m}(A \sin \omega_d t + B \cos \omega_d t) = Ce^{-bt/2m}\sin(\omega_d t + \theta) \tag{2-10}$$

where A, B, C, and $\theta$ are constants which depend on the initial conditions; C and $\theta$ are related to A and B by Equations (2-6) and (2-7); $\omega_d$ is called the damped natural frequency and is given by

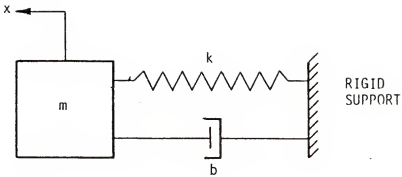



Figure 2-2 Single-degree-of-freedom free vibrational model with viscous damping

$$\omega_d = \sqrt{k/m}\left(1 - \zeta^2\right)^{1/2} = \omega_n\left(1 - \zeta^2\right)^{1/2} \tag{2-11}$$

and is given in units of radians per second. Note that for unidirectional vibration, the damped natural frequency, $\omega_d$, is less than the undamped natural frequency, $\omega_n$. The vibratory response represented by Equation (2-10) is referred to as a damped oscillation. A plot of a typical damped oscillation response is presented in Figure 2-3.

For the case in which the damping is exactly critical, or critically damped ($\zeta = 1$), the solution to Equation (2-8) is of the form

$$x = (A + Bt)e^{-bt/2m} = (A + Bt)e^{-\omega_n t} \tag{2-12}$$

where A and B are constants which depend on the initial conditions. For this case, a typical response plot is presented in Figure 2-4.

For the case in which the damping is greater than critical, or overdamped ($\zeta > 1$), the solution to Equation (2-8) is of the form

$$x = e^{-bt/2m}\left(Ae^{\omega_n\sqrt{\zeta^2 - 1}\ t} + Be^{-\omega_n\sqrt{\zeta^2 - 1}\ t}\right) \tag{2-13}$$

where A and B, as before, are constants which depend on the initial conditions. For this case, a typical response plot is presented in Figure 2-5.

Most practical vibratory systems are underdamped ($\zeta < 1$). Critically damped and overdamped systems, in general, do not exhibit rotor resonance problems. However, the presence of rotor mass unbalance in such systems can result in excessive bearing forces at high

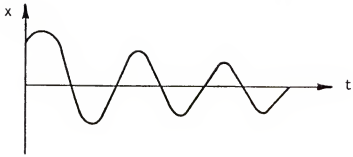



Figure 2-3 Typical damped oscillation response for a single-degree-of-freedom free vibration system with viscous damping and $\zeta < 1$

Figure 2-4 Typical response for single-degree-of-freedom free vibration system with viscous damping and $\zeta = 1$

Figure 2-5 Typical response for single-degree-of-freedom free vibration system with viscous damping and $\zeta > 1$

speeds. Thus, even overdamped systems often require rotor balancing. However, since this is a relatively rare situation, and in such cases rigid rotor balancing is generally sufficient, for the remainder of this study only underdamped systems are considered.

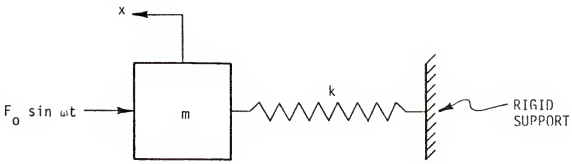
Forced vibration is now considered, beginning with a single-degree-of-freedom undamped system, as illustrated in Figure 2-6. The forcing function is assumed to be sinusoidal and may be realized as a force applied to the mass, shown in Figure 2-6a, or as a prescribed motion of the foundation, shown in Figure 2-6b. The response of the system is defined by the force or motion transmissibility, depending on the form of the forcing function. The force transmissibility is defined as the ratio of the force transmitted to the foundation to the force applied to the mass. The motion transmissibility is defined as the ratio of the motion of the mass to the applied motion of the foundation.

For the case in which a sinusoidal forcing function is applied to the mass, the governing equation becomes

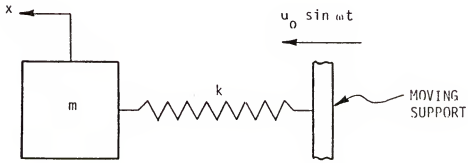
$$m\ddot{x} + kx = F_o \sin \omega t \tag{2-14}$$

where $F_o \sin \omega t$ is the forcing function. Since Equation (2-14) is not homogeneous, it has both a general and a particular solution which together are given by

$$x = A \sin \omega_n t + B \cos \omega_n + \frac{F_o/k}{1 - \omega^2/\omega_n^2} \sin \omega t \tag{2-15}$$




a) FORCE APPLIED TO MASS




b) PRESCRIBED MOTION OF FOUNDATION

Figure 2-6 Single-degree-of-freedom forced vibration model with no damping

where A and B are constants that depend on the initial conditions and $\omega_n$ is, as before, equal to $\sqrt{k/m}$. The first two terms on the right side of Equation (2-15) form the general solution to Equation (2-14). This is identical to the solution for the analogous free vibration case, given by Equation (2-3). In virtually all practical systems there is some damping present which causes the general solution terms (also known as the transient solution) to disappear or damp out with time. However, the particular, or steady-state, solution, given by the last term in Equation (2-15), does not damp out and is, therefore, of primary interest here. This steady-state solution is analogous to synchronous vibration in rotating machinery, as will be shown in the next section.

Considering only the steady-state solution, the force transmissibility, $T_F$, is given by

$$T_F = \frac{kx}{F_o \sin \omega t} = \left(1 - \frac{\omega^2}{\omega_n^2}\right)^{-1} \tag{2-16}$$

where $T_F$ is unbounded when the forcing frequency, $\omega$, is equal to the natural frequency, $\omega_n$. A plot of $T_F$ as a function of the frequency ratio, $\omega/\omega_n$, is presented in Figure 2-7.

For the case in which the foundation has a prescribed sinusoidal motion, the governing equation becomes

$$m\ddot{x} + kx = ku_o \sin \omega t \tag{2-17}$$

where $u_o \sin \omega t$ is the prescribed motion of the support. Equation (2-17) has a solution very similar to Equation (2-15) given by

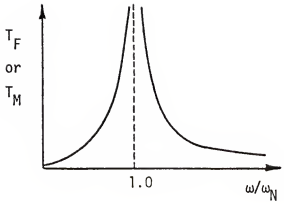



Figure 2-7 Transmissibility of a single-degree-of-freedom forced vibration system with no damping as a function of the frequency ratio

$$x = A \sin \omega_n t + B \cos \omega_n t + \frac{u_o}{1 - \omega^2/\omega_n^2} \sin \omega t \qquad (2\text{-}18)$$

where A and B are constants that depend on the initial conditions and $\omega_n$ is equal to $\sqrt{k/m}$. As for Equation (2-15), the first two terms on the right side of Equation (2-18) form the transient solution, while the last term represents the steady-state solution.

Again considering only the steady-state solution, the motion transmissibility, $T_M$, is given by

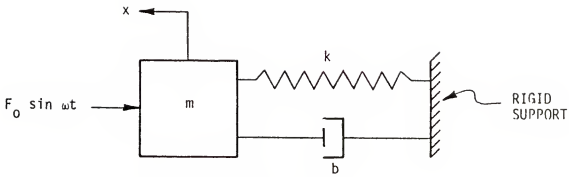
$$T_M = \frac{x}{u_o \sin \omega t} = \left(1 - \omega/\omega_n^2\right)^{-1} \qquad (2\text{-}19)$$

where $T_M$ is unbounded for a frequency ratio of unity. Note that for this model, the force and motion transmissibilities are identical. Thus, the plot in Figure 2-7 applies to motion transmissibility, as well as to force transmissibility.

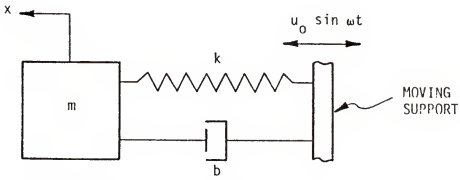
A viscous damping element is now added to the models of Figure 2-6, resulting in a single-degree-of-freedom forced vibration system with viscous damping as illustrated by the models of Figure 2-8. The forcing function, as before, can be represented as either a force applied to the mass (Figure 2-8a) or a prescribed motion of the foundation (Figure 2-8b).

For the case in which a sinusoidal forcing function is applied to the mass, the governing equation becomes

$$m\ddot{x} + b\dot{x} + kx = F_o \sin \omega t \qquad (2\text{-}20)$$




a) FORCE APPLIED TO MASS




b) PRESCRIBED MOTION OF FOUNDATION

Figure 2-8 Single-degree-of-freedom forced vibration model with viscous damping

where $F_o \sin \omega t$ is the applied force. For reasons given above, only the steady-state solution of Equation (2-20) is considered. For positive values of b (in general, b is not negative, although it may be zero), there exists a phase angle between the forcing function $F_o \sin \omega t$ and the steady-state response x. This response is given by

$$x = R \sin (\omega t - \theta) \tag{2-21}$$

where

$$R = \frac{F_o}{k} \left[ \left(1 - \omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2 \right]^{-1/2} \tag{2-22}$$

and

$$\theta = \text{Arctan} \left[ \left(2\zeta\omega/\omega_n\right) \Big/ \left(1 - \omega^2/\omega_n^2\right) \right] \tag{2-23}$$

and $\zeta$ and $\omega_n$ are as defined previously.

The relationship between the force transmitted to the foundation and the applied force is more complex than for the undamped system presented above. For the damped system, both the spring and damper transmit force to the foundation. The total force transmitted to the foundation $F_T$ is given by

$$F_T = b\dot{x} + kx \tag{2-24}$$

where the two terms on the right are ninety degrees out of phase. The ratio of the transmitted force, $F_T$, to the magnitude of the applied force, $F_o$, becomes

$$\frac{F_T}{F_o} = T \sin(\omega t - \psi) \tag{2-25}$$

where

$$T = \sqrt{\frac{\left(1 + 2\zeta\omega/\omega_n\right)^2}{\left(1 - \omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}} \tag{2-26}$$

$$\psi = \text{Arctan}\left[\frac{2\zeta\left(\omega/\omega_n\right)^3}{1 - \left(\omega^2/\omega_n^2\right) + \left(4\zeta^2\omega^2/\omega_n^2\right)}\right] \tag{2-27}$$

and T and $\psi$ are referred to as the transmissibility and the phase angle, respectively, Note that for positive values of $\zeta$, T is bounded for all values of $\omega$. Thus, the existence of damping is necessary to control vibration at a system resonance [compare Equation (2-26) to Equation (2-16)]. Plots of transmissibility, T, and phase angle, $\psi$, are presented as functions of the frequency ratio for several values of $\zeta$ in Figures 2-9 and 2-10, respectively.

The maximum value of the displacement, x, from equation (2-21), occurs for a forcing frequency equal to $[\omega_n(1 - 2\zeta^2)^{1/2}]$. Note that for positive $\zeta$, the maximum steady-state response (in terms of displacement) always occurs for forcing frequencies less than the undamped natural frequency. This is generally true for uni-directional vibratory systems. As will be seen in the next section, this is not true for lateral vibration of rotating machines.

For the case in which the foundation has a prescribed sinusoidal motion (Figure 2-8b), the governing equation becomes

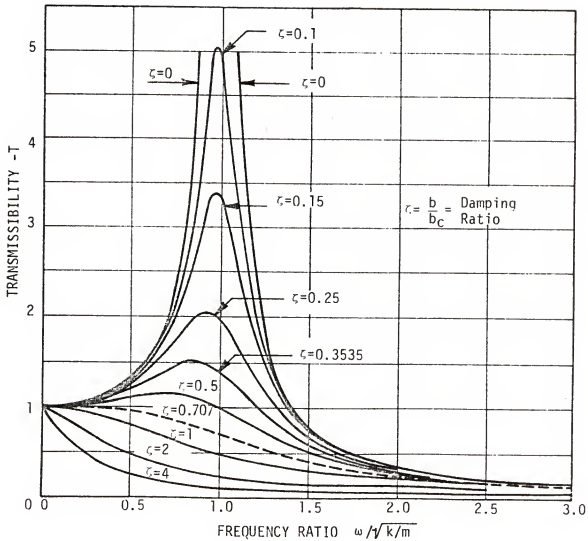



Figure 2-9 Transmissibility of a single-degree-of-freedom forced vibration system with viscous damping as a function of the frequency ratio for several values of the damping ratio

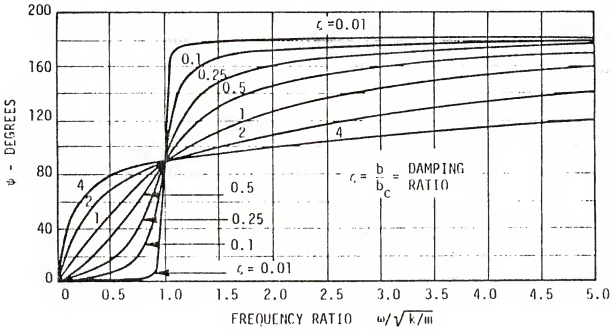



Figure 2-10 Phase angle of a single-degree-of-freedom forced vibration system with viscous damping as a function of the frequency ratio for several values of the damping ratio

$$m\ddot{x} + c\dot{x} + kx = cu_o\omega \cos \omega t + ku_o \sin \omega t \tag{2-28}$$

where $u_o \sin \omega t$ is the prescribed displacement of the foundation and, thus, $u_o\omega \cos \omega t$ is the related prescribed velocity of the foundation (the velocity is the time derivative of the displacement). Again, considering only the steady-state solution to Equation (2-28), the response is given by

$$x = Tu_o \sin (\omega t - \psi) \tag{2-29}$$

where T and $\psi$ are defined by Equations (2-26) and (2-27), respectively. Analogous to Equation (2-25), the motion transmissibility and phase angle are defined from

$$\frac{x}{u_o} = T \sin (\omega t - \psi) \tag{2-30}$$

using Equation (2-29) for x. Thus, the force transmissibility and phase angle are identical to the motion transmissibility and phase angle for the corresponding forms of the forced vibration. Consequently, Figures 2-9 and 2-10 apply to both damped forced vibration cases.

In general, practical vibratory systems have more than one degree of freedom. Practical systems can usually be represented with reasonable accuracy by a number of spring, mass and damper elements, connected together in some fashion, if a sufficient number of elements are used. This is often referred to as a lumped parameter representation. Real systems may exhibit free or forced vibration, or a combination of both. Synchronous lateral vibration of rotating machines,

which is of primary interest in this presentation, is a form of forced vibration where centrifugal forces, due to mass unbalance, provide the forcing function. This is discussed in detail in the following section. A more complete discussion of multiple-degree-of-freedom systems is beyond the scope of this study.

## 2.2 Basic Theory of Lateral Rotor Vibrations

As indicated in the first chapter, rotor vibrations may be in the form of lateral, axial, or torsional vibrations (see Figure 1-1). These vibrations may also be either synchronous or nonsynchronous. As rotor mass unbalance will result in only synchronous lateral vibration, this is the only form of rotor vibration that is of interest in this study. Consequently, the basic rotordynamic theory presented in this section concerns lateral vibration only, with emphasis on the synchronous frequency components.

Early in the twentieth century, H. H. Jeffcott [9] presented an analysis of a simple, concentrated-mass, flexible rotor including the effect of external damping. Although the model is very simple, this analysis demonstrates the effect of the phase shift of the center of gravity of the rotor as its critical speed is traversed. The analysis was proved to be valid and has subsequently formed the basis of later, more detailed, analyses of synchronous vibration in damped rotors. In this section, Jeffcott's analysis is briefly reviewed and similar analyses are performed for simple conical and combined conical-translational whirl. The effect of flexible, anisotropic supports is also discussed.

Jeffcott's original analysis is based on the rotor model presented in Figure 2-11. This model consists of a longitudinally and laterally symmetric rotor, with a single concentrated mass in the center, supported by a flexible, massless shaft which is in turn mounted on simple supports. This analysis can be applied to the rotor model presented in Figure 2-12, with no modifications whatsoever, so long as the values of the model parameters (mass, stiffness, damping, and eccentricity) are the same for both models.

An end view of the rotor model is presented in Figure 2-13, where the origin of the stationary (non-rotating) coordinate system XY is located at the undeflected centerline of the rotor. For a vertical rotor, the undeflected centerline is coincident with the line of bearing centers. For a horizontal rotor, these two lines differ by the static gravity sag of the rotor. The point E represents the deflected centerline of the rotor, while the point M represents the location of the center of mass of the rotor. Since the coordinate system is not rotating, the sketch in Figure 2-13 represents an instant of time when the rotor is at an angle $\omega t$ from the X axis. Rotation of the rotor is counter-clockwise. The coordinates of the point E at that instant in time are x and y. The coordinates of the point M at the same instant in time are x' and y' related to x and y by

$$x' = x + a \cos \omega t \tag{2-31}$$

$$y' = y + a \sin \omega t \tag{2-32}$$

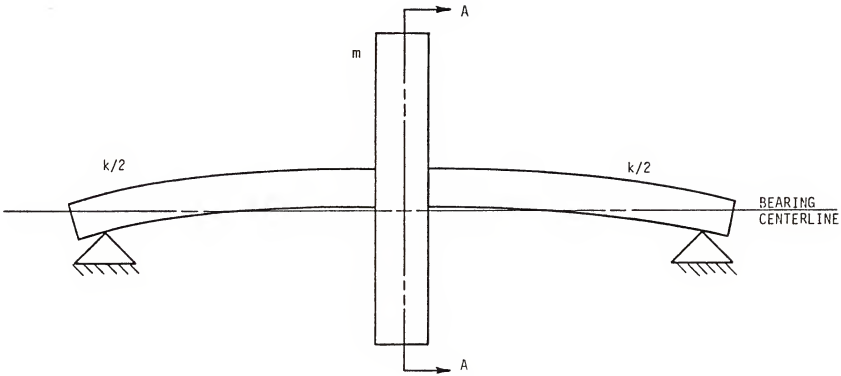



Figure 2-11 Rotor model used for original Jeffcott analysis

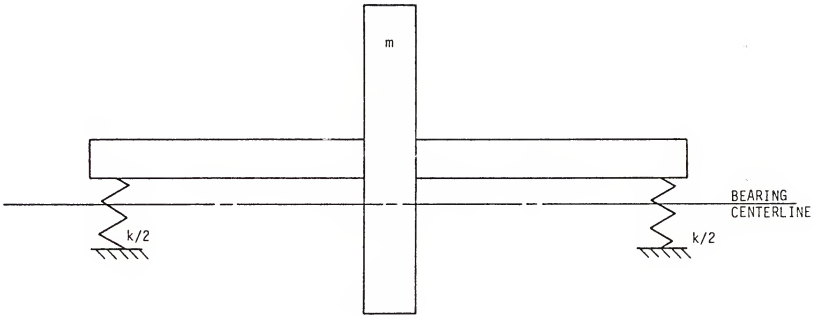



Figure 2-12 Analogous rotor model for original Jeffcott analysis

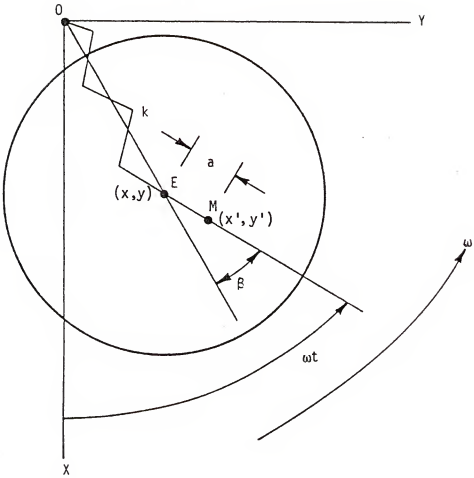



Figure 2-13 Cross-sectional view of Jeffcott rotor model (section A-A)

where a is the fixed distance between points E and M. This fixed distance is known as the mass eccentricity. The differential equations of motion for the rotor are given by

$$m\ddot{x}' + b\dot{x} + kx = 0 \tag{2-33}$$

$$m\ddot{y}' + b\dot{y} + ky = 0 \tag{2-34}$$

where m is the mass of the rotor and b and k are the external linear viscous damping and stiffness, respectively. The dots again represent derivatives taken with respect to time. In the X-direction, the D'Alembert force, $m\ddot{x}'$, is proportional to the acceleration in the X-direction of the point M. However, the damping and stiffness act at the point E and, therefore, result in forces proportional to $\dot{x}$ and x, respectively. The forces in the Y-direction act in a smaller fashion. Substituting Equations (2-31) and (2-32) into Equations (2-33) and (2-34), to eliminate x' and y', and rearranging terms gives Equations (2-35) and (2-36), for $\omega$ is not a function of time.

$$m\ddot{x} + b\dot{x} + kx = ma\omega^2 \cos \omega t \tag{2-35}$$

$$m\ddot{y} + b\dot{y} + ky = ma\omega^2 \sin \omega t \tag{2-36}$$

Note that at this point the rotor system is assumed to be symmetric with respect to the XY coordinate system. That is, the system parameters (damping and stiffness) are assumed to have equal values in the X and Y directions. The effect of anisotropic supports is considered later.

There are various ways to solve Equations (2-35) and (2-36), including the use of Laplace transforms. However, the simplest method

of solution is to plug an assumed solution into the differential equations and solve for the constants, since the form of the solution is obvious from Figure 2-13. Jeffcott solves these equations for both the transient (general) and steady-state (particular) solutions. However, since no sources of instability are present in this model (assuming positive damping), the transient (nonsynchronous) solutions will necessarily decay to zero. Therefore, only the steady-state (synchronous) solutions are of interest here. From Figure 2-13, it is clear that

$$x = A \cos (\omega t - \beta) \tag{2-37}$$

$$y = A \sin (\omega t - \beta) \tag{2-38}$$

where A is the distance from O to E and $\beta$ is known as the phase angle. Substituting Equations (2-37) and (2-38) and their derivatives into Equations (2-35) and (2-36) and solving for the constants A and $\beta$ gives

$$A = \frac{ma\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + b^2\omega^2}} = \frac{a\left(\omega/\omega_n\right)^2}{\sqrt{\left(1 - \omega^2/\omega_n^2\right)^2 + 4\left(\zeta\omega/\omega_n\right)^2}} \tag{2-39}$$

$$\beta = \arctan \frac{b\omega}{k - m\omega^2} = \arctan \left(\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2}\right) \text{ where } 0 \leqslant \beta \leqslant \pi \tag{3-40}$$

where $\zeta$ and $\omega_n$ are the same as for uni-directional vibration. The critical speed of the rotor is defined as the speed at which the value

of A is a maximum. The equation for the critical speed, $\omega_{CR}$, is derived by differentiating the equation for A with respect to $\omega$, setting it equal to zero, and solving for $\omega$, resulting in

$$\omega_{CR} = \frac{2k}{\sqrt{4mk - 2b^2}} = \frac{\omega_n}{\sqrt{1 - 2\zeta^2}} \tag{2-41}$$

It is apparent from Equation (2-41) that the effect of increasing damping is to increase the critical speed.

A similar analysis can be conducted for a rotor subjected to a conical mode of vibration. This analysis is performed for the rotor model illustrated in Figure 2-14. This rotor model is longitudinally symmetric. That is, a centrally located disk is supported by bearings of equal stiffness. The disk has a mass, m, and polar and transverse moments of inertia, $I_p$ and $I_t$, respectively. The analysis can also be applied to the rotor model presented in Figure 2-15, with no modifications whatsoever, so long as the values of the model parameters are the same for both models.

A view of the rotor in Figure 2-14 at section A-A is presented in Figure 2-16 in the XY coordinate system defined above. The mass axis of the rotor is assumed to be displaced angularly with respect to the undeflected shaft axis, with no eccentricity of the center of gravity. Assuming that the mass shift angle, $\theta_a$, is small, the distance from the undeflected shaft axis (the point E in Figure 2-16) to the mass axis (the point M in Figure 2-16) at any cross-section of the rotor is, equal to $\theta_a z$, where z is the axial distance from the center of gravity

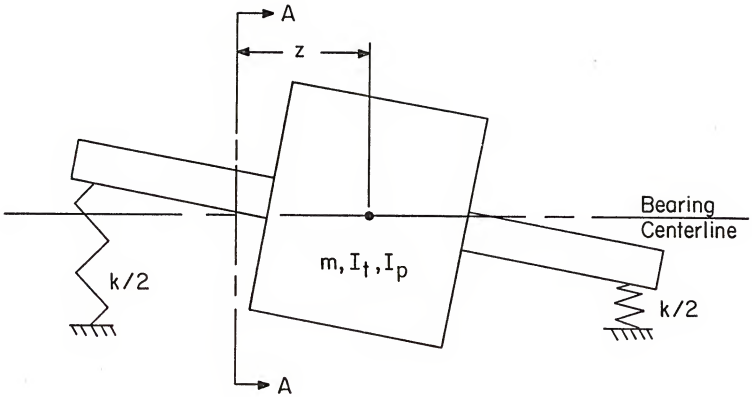



Figure 2-14 Rotor model for conical Jeffcott analysis

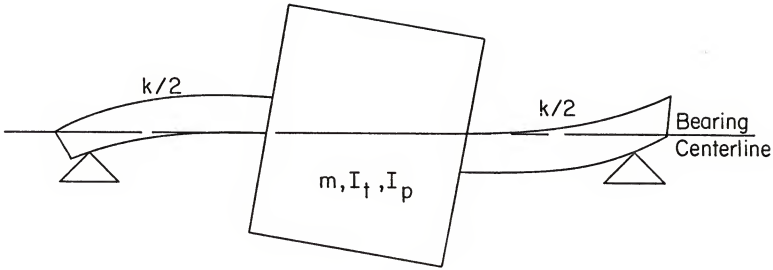



Figure 2-15 Analogous rotor model for conical Jeffcott analysis

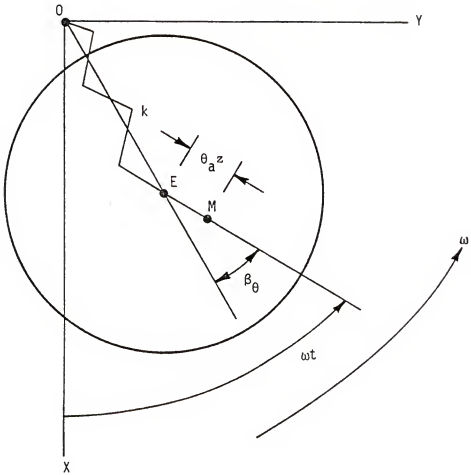



Figure 2-16 Cross-sectional view of conical Jeffcott rotor model (section A-A)

of the rotor to the cross-section, Then $\theta_x$ and $\theta_y$ represent the X and Y components of $\theta$, the angle between the undeflected shaft axis and the bearing axis. Consequently x, y, x' and y' in Figure 2-16 are equivalent to the expressions

$$x = \theta_x z \tag{2-42}$$

$$y = \theta_y z \tag{2-43}$$

$$x' = \theta_x z + \theta_a z \cos \omega t \tag{2-44}$$

and

$$y' = \theta_y z + \theta_a z \sin \omega t \tag{2-45}$$

The angle between the mass axis of the rotor and the bearing axis, $\theta'$, may then be represented as having X and Y components

$$\theta'_x = \frac{x'}{z} = \theta_x + \theta_a \cos \omega t \tag{2-46}$$

$$\theta'_y = \frac{y'}{z} = \theta_y + \theta_a \sin \omega t \tag{2-47}$$

The differential equations of motion of the rotor are then including the "gyroscopic effect" [10] represented by Equations (2-48) and (2-49).

$$I_t \ddot{\theta}'_x + I_p \omega \dot{\theta}'_y + b_\theta \dot{\theta}_x + k_\theta \theta_x = 0 \tag{2-48}$$

$$I_t \ddot{\theta}'_y - I_p \omega \dot{\theta}'_x + b_\theta \dot{\theta}_y + k_\theta \theta_y = 0 \tag{2-49}$$

where $b_\theta$ and $k_\theta$ are the angular viscous damping and angular stiffness coefficients, respectively, for the symmetrically supported rotor. Substituting for the primed variables from Equations (2-46) and (2-47), taking the time derivatives and collecting terms results in Equations (2-50) and (2-51).

$$I_t\ddot{\theta}_x + I_p\omega\dot{\theta}_y + b_\theta\dot{\theta}_x + k_\theta\theta_x = (I_t - I_p)\,\theta_a\omega^2 \cos \omega t \quad (2\text{-}50)$$

$$I_t\ddot{\theta}_y - I_p\omega\dot{\theta}_x + b_\theta\dot{\theta}_y + k_\theta\theta_y = (I_t - I_p)\,\theta_a\omega^2 \sin \omega t \quad (2\text{-}51)$$

Note that these two differential equations are coupled due to the gyroscopic terms. However, it is possible to decouple these equations. Note that $\theta_x$ and $\theta_y$ take the form of Equations (2-52) and (2-53)

$$\theta_x = B \cos (\omega t - \beta_\theta) \quad (2\text{-}52)$$

$$\theta_y = B \sin (\omega t - \beta_\theta) \quad (2\text{-}53)$$

where B represents the angle $\theta$. Their first two time derivatives are simply

$$\dot{\theta}_x = -B\omega \sin (\omega t - \beta_\theta) \quad (2\text{-}54)$$

$$\dot{\theta}_y = B\omega \cos (\omega t - \beta_\theta) \quad (2\text{-}55)$$

$$\ddot{\theta}_x = -B\omega^2 \cos (\omega t - \beta_\theta) \quad (2\text{-}56)$$

$$\ddot{\theta}_y = -B\omega^2 \sin (\omega t - \beta_\theta) \quad (2\text{-}57)$$

These equations lead to the equalities

$$\dot{\theta}_y\omega = -\ddot{\theta}_x \tag{2-58}$$

$$\dot{\theta}_x\omega = \ddot{\theta}_y \tag{2-59}$$

Substituting Equations (2-58) and (2-59) into Equations (2-51) and (2-52) results in Equations (2-60) and (2-61), which are completely decoupled from one another.

$$(I_t - I_p)\ddot{\theta}_x + b_\theta\dot{\theta}_x + k_\theta\theta_x = (I_t - I_p)\theta_a\omega^2 \cos \omega t \tag{2-60}$$

$$(I_t - I_p)\ddot{\theta}_y + b_\theta\dot{\theta}_y + k_\theta\theta_y = (I_t - I_p)\theta_a\omega^2 \sin \omega t \tag{2-61}$$

Equations (2-60) and (2-61) are directly analogous to Equations (2-35) and (2-36) with the substitutions of the angular coefficients and variables. Consequently, the same substitutions may be made in Equations (2-39), (2-40), and (2-41) in order to obtain Equations (2-62) through (2-64), which together with Equations (2-52) and (2-53) define the conical mode of vibration of the rotor model in Figure 2-14.

$$B = \frac{(I_t - I_p)\theta_a\omega^2}{\sqrt{\left[k_\theta - (I_t - I_p)\omega^2\right]^2 + b_\theta^2\omega^2}}$$

$$= \frac{\theta_a(\omega/\omega_{n\theta})^2}{\sqrt{\left(1 - \omega^2/\omega_{n\theta}^2\right)^2 + 4\left(\zeta_\theta\omega/\omega_{n\theta}\right)^2}} \tag{2-62}$$

$$\beta_\theta = \arctan \frac{b_\theta \omega}{k_\theta - (I_t - I_p)\omega^2}$$

$$= \arctan \frac{2\zeta_\theta \omega/\omega_{n\theta}}{1 - \omega^2/\omega_{n\theta}^2} \quad \text{where } 0 \leqslant \beta_\theta \leqslant \pi \tag{2-63}$$

$$\omega_{CR} = \frac{2k_\theta}{\sqrt{4(I_t - I_p)k_\theta - 2b_\theta^2}} = \frac{\omega_{n\theta}}{\sqrt{1 - 2\zeta_\theta^2}} \tag{2-64}$$

where $\omega_{n\theta}$ and $\zeta_\theta$ are defined the same as $\omega_n$ and $\zeta$, except that $(I_t - I_p)$ is substituted for m, $k_\theta$ for k, and $b_\theta$ for b. Note that both the damping and gyroscopic terms tend to raise the critical speed. This rotor model will also exhibit a separate purely translational critical speed, whose response may be calculated using Jeffcott's original analysis, as described above.

A rotor, similar to that in Figure 2-15 but longitudinally nonsymetric, will not exhibit purely translatory and conical modes of vibration. Instead, this rotor will have two rigid-body critical speeds whose mode shapes are each a combination of translatory and conical whirl. Generally speaking, the translatory whirl predominates for one of these mode shapes while the conical whirl predominates for the other. Such a rotor model is presented in Figure 2-17. The governing differential equations of motion for this model are presented in Equations (2-65) through (2-68).

$$m\ddot{x}' + b\dot{x} + kx + (b_2\ell_2 - b_1\ell_1)\dot{\theta}_x + (k_2\ell_2 - k_1\ell_1)\theta_x = 0 \tag{2-65}$$

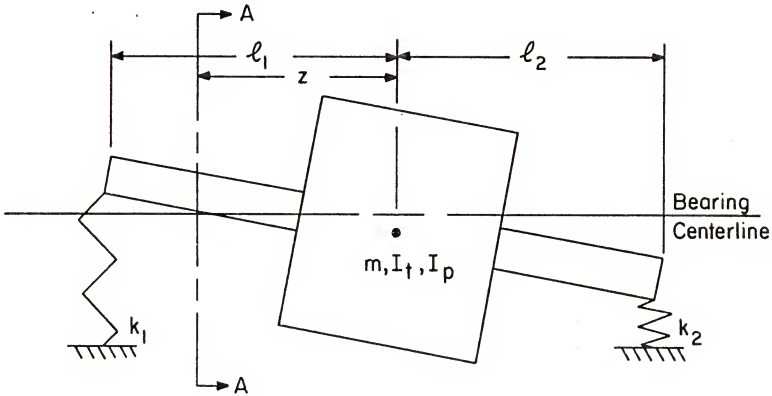



Figure 2-17 Rotor model for coupled translational-conical Jeffcott analysis

$$I_t\ddot{\theta}_x' + I_p\omega\dot{\theta}_y' + b_\theta\dot{\theta}_x + k_\theta\theta_x + (b_2\ell_2 - b_1\ell_1)\dot{x}$$

$$+ (k_2\ell_2 - k_1\ell_1)x = 0 \tag{2-66}$$

$$m\ddot{y}' + b\dot{y} + ky + (b_2\ell_2 - b_1\ell_1)\dot{\theta}_y + (k_2\ell_2 - k_1\ell_1)\theta_y = 0 \tag{2-67}$$

$$I_t\ddot{\theta}_y' - I_p\omega\dot{\theta}_x' + b_2\dot{\theta}_y + k_\theta\theta_y + (b_2\ell_2 - b_1\ell_1)\dot{y}$$

$$+ (k_2\ell_2 - k_1\ell_1)y = 0 \tag{2-68}$$

These equations are derived using superposition of the translatory and conical unbalance distributions and modes of vibration, where the total resultant vibration is a linear combination of pure translatory and pure conical motion. The non-primed independent variables in equations (2-65) through (2-68) are represented by

$$x = A \cos (\omega t - \beta_T) \tag{2-69}$$

$$y = A \sin (\omega t - \beta_T) \tag{2-70}$$

$$\theta_x = B \cos (\omega t - \beta_C) \tag{2-71}$$

$$\theta_y = B \sin (\omega t - \beta_C) \tag{2-72}$$

where $\beta_T$ and $\beta_C$ represent $\beta$ in Figures 2-13 and 2-16, respectively. The coefficients A and B represent the distance OE and the angle $\theta$, respectively, from the same figures.

Equations (2-65) and (2-66) may be decoupled from Equations (2-67) and (2-68) using the substitutions given by Equations (2-58) and (2-59) as described above. However, Equations (2-65) and (2-67) cannot be decoupled from Equations (2-66) and (2-68) unless the terms in parentheses go to zero. If the conditions in Equations (2-73) and (2-74) are satisfied,

$$b_1\ell_1 = b_2\ell_2 \tag{2-73}$$

$$k_1\ell_1 = k_2\ell_2 \tag{2-74}$$

then Equations (2-65) and (2-66) decouple and reduce to Equations (2-33) and (2-48), indicating the presence of separate pure translatory and conical modes of vibration. Equations (2-67) and (2-68) are similarly affected. If the conditions in either or both of Equations (2-73) and (2-74) are not satisfied, then Equations (2-65) and (2-67) do not decouple from Equations (2-66) and (2-68), respectively. In either case, Equations (2-65) and (2-66) are decoupled from and analogous to Equations (2-67) and (2-68) and solving the first two equations implies a solution to the latter two equations. Therefore, only Equations (2-65) and (2-66) are solved, from which the solutions of Equations (2-67) and (2-68) are inferred.

Substitutions for $x'$ and $\theta'_x$ from Equations (2-31) and (2-46) and the substitutions from Equations (2-58) and (2-59) are introduced into Equations (2-65) and (2-66) and the terms are rearranged resulting in Equations (2-75) and (2-76).

$$m\ddot{x} + b\dot{x} + kx + (b_2\ell_2 - b_1\ell_1)\dot{\theta}_x$$

$$+ (k_2\ell_2 - k_1\ell_1)\theta_x = ma\omega^2 \cos \omega t \tag{2-75}$$

$$(I_t - I_p)\ddot{\theta}_x + b_\theta\theta_x + k_\theta\theta_x + (b_2\ell_2 - b_1\ell_1)\dot{x}$$

$$+ (k_2\ell_2 - k_1\ell_1)x = (I_t - I_p)\theta_a\omega^2 \cos \omega t \tag{2-76}$$

The substitutions in Equations (2-77) through (2-80) are implied in Equations (2-75) and (2-76) based on the rotor model in Figure 2-17.

$$b = b_1 + b_2 \tag{2-77}$$

$$k = k_1 + k_2 \tag{2-78}$$

$$b_\theta = b_1\ell_1^2 + b_2\ell_2^2 \tag{2-79}$$

$$k_\theta = k_1\ell_1^2 + k_2\ell_2^2 \tag{2-80}$$

In addition, the substitutions in Equations (2-81) through (2-83) are made in order to reduce Equations (2-75) and (2-76) to Equations (2-84) and (2-85).

$$I = I_t - I_p \tag{2-81}$$

$$b_{21} = (b_2\ell_2 - b_1\ell_1)/\ell \tag{2-82}$$

$$k_{21} = (k_2\ell_2 - k_1\ell_1)/\ell \tag{2-83}$$

$$m\ddot{x} + b\dot{x} + kx + b_{21}\ell\dot{\theta}_x + k_{21}\ell\theta_x = ma\omega^2 \cos \omega t \tag{2-84}$$

$$I\ddot{\theta} + b_\theta\dot{\theta}_x + k_\theta\theta_x + b_{21}\ell\dot{x} + k_{21}\ell x = I\theta_a\omega^2 \cos \omega t \tag{2-85}$$

Direct substitution of Equations (2-69) and (2-71) into Equations (2-84) and (2-85) results in four simultaneous transcendental equations for which there is no apparent closed-form solution. However, Equations (2-69) and (2-71) can be rewritten as

$$x = G \sin \omega t + D \cos \omega t \tag{2-86}$$

$$\theta_x = E \sin \omega t + F \cos \omega t \tag{2-87}$$

where

$$G = A \sin \beta_T \tag{2-88}$$

$$D = A \cos \beta_T \tag{2-89}$$

$$E = B \sin \beta_C \tag{2-90}$$

$$F = B \cos \beta_C \tag{2-91}$$

Substituting Equations (2-86) and (2-87) into Equations (2-84) and (2-85) results in Equations (2-92) through (2-95), a set of four simultaneous linear equations in G, D, E, and F.

$$(k - m\omega^2)G - b\omega D + k_{21}\ell E - b_{21}\ell\omega F = 0 \tag{2-92}$$

$$b\omega G + (k - m\omega^2)D + b_{21}\ell\omega E + k_{21}\ell F = ma\omega^2 \tag{2-93}$$

$$k_{21}\ell G - b_{21}\ell\omega D + (k_\theta - I\omega^2)E - b_\theta\omega F = 0 \tag{2-94}$$

$$b_{21}\ell G + k_{21}\ell D + b_\theta\omega E + (k_\theta - I\omega^2)F = I\theta_a\omega^2 \tag{2-95}$$

The expressions for the coefficients G, D, E, and F which satisfy Equations (2-92) through (2-95) are presented in Equations (2-96) through (2-99),

$$G = \{ma\omega^2[b_\theta\omega(S_1 + b_\theta b\omega^2) + k_{\theta I}S_\theta] + I\theta_a\omega^2[b_{21}\ell\omega(T_1 + T_2) - k_{21}\ell T_3]\}/H \tag{2-96}$$

$$D = \{ma\omega^2[b_\theta\omega S_x + k_{\theta I}(k_{xm}k_{\theta I} - S_1)] + I\theta_a\omega^2[-b_{21}\ell\omega T_3 + k_{21}\ell(T_1 - T_2)]\}/H \tag{2-97}$$

$$E = \{ma\omega^2[b_{21}\ell\omega(T_1 + T_2) - k_{21}\ell T_3] + I\theta_a\omega^2[b\omega(S_1 + b_\theta b\omega^2) + k_{xm}S_x]\}/H \tag{2-98}$$

$$F = \{ma\omega^2[-b_{21}\ell\omega T_3 + k_{21}\ell(T_1 - T_2)] + I\theta_a\omega^2[b\omega S_\theta + kxm\ (k_{xm}k_{\theta I} - S_1)]\}/H \tag{2-99}$$

where

$$S_1 = k_{21}^2\ell^2 - b_{21}^2\ell^2\omega^2 \tag{2-100}$$

$$S_\theta = k_{\theta I}b\omega - 2k_{21}b_{21}\ell^2\omega \tag{2-101}$$

$$S_x = k_{xm} b_\theta \omega - 2k_{21} b_{21} \ell^2 \omega \tag{2-102}$$

$$T_1 = k_{21}^2 \ell^2 + b_{21}^2 \ell^2 \omega^2 \tag{2-103}$$

$$T_2 = k_{xm} k_{\theta I} - b b_\theta \omega^2 \tag{2-104}$$

$$T_3 = k_{xm} b_\theta \omega + k_{\theta I} b \omega \tag{2-105}$$

$$H = (k_{xm}^2 + b^2\omega^2)(k_{\theta I}^2 + b_\theta^2 \omega^2)$$

$$+ T_1^2 - 2T_2 S_1 - 4T_3 b_{21} k_{21} \ell^2 \omega \tag{2-106}$$

$$k_{xm} = k - m\omega^2 \tag{2-107}$$

$$k_{\theta I} = k_\theta - I\omega^2 \tag{2-108}$$

It is then possible to derive x and $\theta_x$ in the form of Equations (2-69) and (2-71) by using the relations

$$A = \sqrt{G^2 + D^2} \tag{2-109}$$

$$\beta_T = \arctan (G/D) \tag{2-110}$$

$$B = \sqrt{E^2 + F^2} \tag{2-111}$$

$$\beta_C = \arctan (E/F) \tag{2-112}$$

where

$$0 \leqslant \beta_T \leqslant \pi \text{ and } 0 \leqslant \beta_C \leqslant \pi$$

The resulting expressions for A and B may then be differentiated with respect to $\omega$, in order to calculate the critical speeds of the rotor, as described above. The algebraic manipulation required to apply Equations (2-109) through (2-112) and calculate the critical speeds is quite involved. For rotors where the damping is expected to be very small the error in the predicted values of critical speed introduced by neglecting damping is also very small. In light of the fact that the values of rotor mass, inertia and support stiffness used for these calculations are often only approximate, the error mentioned above becomes even less significant. Neglecting damping simplifies Equations (2-96) through (2-106) tremendously, and Equations (2-109) through (2-112) become

$$A = [ma\omega^2 k_{\theta I} - I\theta_a \omega^2 k_{21} \ell]/[k_{xm} k_{\theta I} - k_{21}^2 \ell^2] \tag{2-113}$$

$$\beta_T = 0 \text{ or } \pi \text{ (below or above } \omega_{CR}) \tag{2-114}$$

$$B = [I\theta_a \omega^2 k_{xm} - ma\omega^2 k_{21} \ell]/[k_{xm} k_{\theta I} - k_{21}^2 \ell^2] \tag{2-115}$$

$$\beta_C = 0 \text{ or } \pi \text{ (below or above } \omega_{CR}) \tag{2-116}$$

For an undamped rotor, at the critical speed the amplitude of vibration becomes infinite. This occurs when the denominators of Equations (2-113) and (2-115) are equal to zero, or when

$$k_{xm} k_{\theta I} - k_{21}^2 \ell^2 = 0 \tag{2-117}$$

Obviously, if $k_{21}$ is equal to zero, a critical speed occurs where $k_{xm}$ or $k_{\theta I}$ is equal to zero, which represents the decoupled mode case discussed above. Solving Equation (2-117) for $\omega_{CR}^2$, for $k_{21}$ not equal to zero, gives the two solutions

$$\omega_{CR}^2 = \frac{k}{2m} + \frac{k_\theta}{2I} \pm \sqrt{\left(\frac{k}{2m} - \frac{k_\theta}{2I}\right)^2 + \frac{k_{21}^2 \ell^2}{Im}} \tag{2-118}$$

where only the positive square roots of $\omega_{CR}^2$ are of interest. Equation (2-118) may be rewritten as Equation (2-119) from which it becomes clear that $\omega_{CR}^2$ must be real and positive.

$$\omega_{CR}^2 = \frac{k}{2m} + \frac{k_\theta}{2I} \pm \sqrt{\left(\frac{k}{2m} + \frac{k_\theta}{2I}\right)^2 - \frac{k_1 k_2 \ell^2}{mI}} \tag{2-119}$$

Regardless of the complexity of its geometric configuration, a rigid rotor can be represented exactly by a mass, transverse and polar moments of inertia and a center of gravity location. Thus, the mass unbalance of a rigid rotor can also be represented exactly by two parameters, a and $\theta_a$. Since a can be reduced to zero with a single correction mass located at an arbitrary axial location and $\theta_a$ can be reduced to zero, with an appropriate pair of correction masses at appropriate axial locations, it seems reasonable to assume that any arbitrary unbalance in a rigid rotor can be corrected with a maximum of three correction masses. In fact, since the correction mass for a can be located in the same diametral plane as one of the correction masses for $\theta_a$ and the two correction masses vectorally summed to form

a single correction mass, no more than two correction masses are actually required to balance a rigid motor.

A flexible rotor, however, generally has more than two degrees of freedom and often requires more than two correction masses. In essence, when a rotor bends, the mass centerline also bends and the mass unbalance can no longer be represented by a and $\theta_a$. Thus, the rigid rotor corrections which have been made to reduce a and $\theta_a$ to zero, do not cancel the new mass unbalance distribution that occurs when the rotor bends.

In the linear analysis of rotating machines, two assumptions are made which are central to rotor balancing: (1) Synchronous rotor response is linearly related to mass unbalance, and (2) the effect of individual unbalances may be superposed. These assumptions, while not strictly adhered to in any real nonlinear system, are generally very accurate for relatively small changes in unbalance.

A complex notation is often used to present the in-phase and out-phase components of rotordynamic response. The use of this complex notation simplifies the modeling and analytical procedures for realistic rotor systems and is convenient when considering rotor mass unbalance. Most rotor balancing procedures employ some form of complex notation.

Rotor response may be represented using either a stationary or rotating frame of reference, as illustrated in Figure 2-18. The stationary coordinate system, XY, is fixed in space, while the rotating coordinate system, UV, is fixed to the bearing centerline and rotates

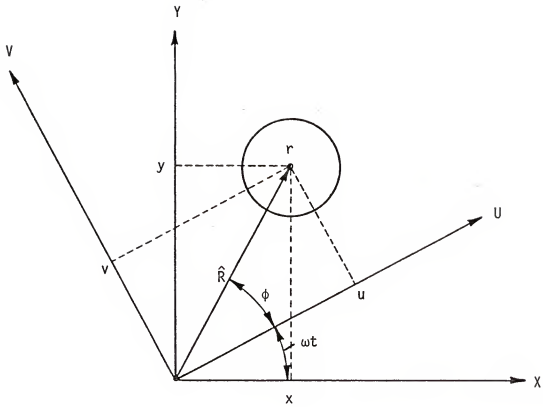



Figure 2-18 Stationary and rotating coordinate systems for measurement of synchronous rotor vibration

with the rotor. For a constant speed of rotation and synchronous vibration only (the condition of interest for rotor balancing), the location of the rotor axis in the rotating coordinate system, represented by the vector $\hat{R}$ in Figure 2-18, does not change with time if the rotor supports are symmetric. This is referred to as a circular orbit. This vector may be specified by either the orthogonal components u and v, or the magnitude r and angle $\phi$, where

$$r = \sqrt{u^2 + v^2} \tag{2-120}$$

$$\phi = \text{Arg}\ (u + iv) \tag{2-121}$$

The two-dimensional vector $\hat{R}$ may be represented in complex notation by

$$\hat{R} = u + iv = re^{i\phi} \tag{2-122}$$

where

$$e^{i\phi} = \cos\ \phi + i\ \sin\ \phi \tag{2-123}$$

$$i = \sqrt{-1} \tag{2-124}$$

In standard nomenclature, u and v are referred to as the real and imaginary components of $\hat{R}$, and r and $\phi$ are referred to as the magnitude and phase angle of $\hat{R}$, respectively. In essence, u and v may be thought of as cartesian coordinates while r and $\phi$ are thought of as polar coordinates. The superior caret is used to indicate complex variables.

The vector $\hat{R}$ is represented in the stationary coordinate system in a similar form as Equation (2-122) by

$$\hat{R} = re^{i\gamma} \tag{2-125}$$

where

$$\gamma = \omega t + \phi \tag{2-126}$$

for a leading phase angle convention, as discussed below. Equation (2-125) may also be written in the form

$$\hat{R} = re^{i\phi}e^{i\omega t} = (u + iv)e^{i\omega t} \tag{2-127}$$

Equation (2-127) may be further expanded into either of the following forms,

$$\hat{R} = (u \cos \omega t - v \sin \omega t) + i(u \sin \omega t + v \cos \omega t) \tag{2-128}$$

or

$$\hat{R} = r \cos (\omega t + \phi) + ir \sin (\omega t + \phi) \tag{2-129}$$

The real and imaginary components of Equations (2-128) and (2-129) represent the time dependent values of x and y of Figure 2-18, respectively.

In general, actual rotor vibration is measured using a vibration sensor which is fixed in space and is, by common convention, located in the X-direction. Thus, the measured vibration, x, can be represented by the real part of Equation (2-128) or (2-129).

$$x = u \cos \omega t - v \sin \omega t = r \cos (\omega t + \phi) \tag{2-130}$$

The real part of Equation (2-129) is identical in form to Equation (2-37) from the rotordynamic analysis, except for the sign of the phase angle, $\phi$ (or $\beta$). This difference in sign occurs because $\beta$ is taken to be positive in a direction opposite to rotation (referred to as a lagging phase angle) while $\phi$ is taken to be positive in the same direction as rotation (referred to as a leading phase angle). For the remainder of this presentation, leading phase angles are assumed. However, so long as the phase angles and the balance mass angles are taken to be positive in the same direction, the balancing calculations for most balancing methods, including the Unified Balancing Approach, are identical for leading and lagging phase angles.

The usual convention is to write the synchronous rotor response, x, as

$$\hat{x} = x_c + ix_s = re^{i\phi} \tag{2-131}$$

where $x_c$ and $x_s$ are used to represent u and v, respectively, and Equation (2-131) is actually a shorthand form for

$$x = \text{Real}\ [(x_c + ix_s)e^{i\omega t}] = \text{Real}\ [re^{i\phi}e^{i\omega t}] \tag{2-132}$$

Thus, synchronous rotor vibration is generally represented using the complex notation of Equation (2-131) in either cartesian or polar form. This convention is followed for the remainder of this presentation. The locations and sizes of discrete mass unbalances are identified by

$$\hat{u} = u_c + iu_s = r_u e^{i\phi_u} \tag{2-133}$$

where $u_c$ and $u_s$ are the components in the U and V directions and $r_u$ and $\phi_u$ are the magnitude and phase angle relative to the UV coordinate system, respectively. The form of Equation (2-133) is exactly analogous to Equation (2-131).

If a vibration sensor is also located in the Y-direction of Figure 2-18, the vibration which is measured is represented by the imaginary part of Equation (2-128) or (2-129) for the circular orbit case (symmetric rotor supports). In the notation of Equation (2-131), $\hat{y}$ is given by

$$\hat{y} = y_c + iy_s = r_y e^{i\phi_y} \tag{2-134}$$

where for a circular orbit

$$y_c = x_s \tag{2-135}$$

$$y_s = -x_c \tag{2-136}$$

$$r_y = r \tag{2-137}$$

$$\phi_y = \phi + \pi/2 \tag{2-138}$$

and the phase angles are measured in radians.

For the more complicated situation of anisotropic bearing supports (elliptical orbit case), Equations (2-131) and (2-134) remain valid, but the relations in equations (2-135) through (2-138) no longer hold. Thus, the notation and calculations required for balancing a rotor with an elliptical orbit is not different than that for a

circular orbit, when using x and y vibration measurements individually. However, treating the x and y vibration measurements as independent data is not entirely accurate and can lead to difficulties when attempting to reduce the two simultaneously in a single balancing run. It is preferable to combine the two vibration measurements into a single independent data sample. Lund [10] has shown that an elliptical orbit of this type is due to a combination of forward and backward precessing synchronous whirl. Forward precession indicates that the rotor whirl is occurring in the same direction as the shaft rotation. Conversely, backward precession indicates the direction of whirl is opposite that of rotation. Thus, an elliptical orbit is actually formed from two circular orbits precessing in opposite directions. Since rotor unbalance does no work on the backward precession component [10], only the forward precession component is of interest in balancing. Based on a suggestion from Lund [11], a method has been developed for extracting the forward precession component of the synchronous rotor whirl from a pair of vibration readings at a single axial plane. This is done by subtracting from one of the vibration vectors an angle equal to the angle from the corresponding vibration sensor to the other sensor measured in the direction of shaft rotation (for a leading phase angle). The resulting vector is then vector averaged (formed by half the vector sum) with the other vibration vector to give the forward precession component of the synchronous vibration. While this may be done for any sensor separation angle (other than 0 and 180 degrees), the best results are usually obtained with sensor separation angles in the

neighborhood of 90 degrees. This feature has been included in the general Unified Balancing Approach procedure (apparently this is the first time such a forward precession extraction method has been used in balancing calculations).

# CHAPTER 3
# REVIEW OF LITERATURE ON ROTOR BALANCING

While the necessity of rotor balancing was demonstrated by Jeffcott in his classic paper [9], published in 1919, the first significant contributions to the rotor balancing literature did not appear until almost 1930. Prior to the 1950s, the balancing literature was limited to balancing of rigid, and a few very simple flexible, rotors. There is no documentation during this period of any systematic balancing procedure using more than two balancing planes. The first flexible rotors of significance to be built were steam turbine rotors. Initially, these rotors were balanced using simple, rigid-rotor procedures. However, as these rotors became more flexible, and were often operated supercritically, the existing balancing procedures became inadequate. This led to the development of a number of balancing methods specifically designed for flexible rotors.

Flexible rotor balancing procedures can generally be divided into two groups, modal balancing and influence coefficient balancing (as defined in Chapter 1), of which modal balancing was developed first. The development of modal balancing began in the early 1950s. As very limited instrumentation and computational tools were available at that time, a balancing method was needed that did not depend heavily on such tools. Modal balancing fit naturally into these requirements as only simple calculations are required and operator

insight is the primary ingredient, rather than large quantities (and quality) of vibration data.

Influence coefficient balancing was developed some years later, made possible by improvements in instrumentation and the introduction of the digital computer. Consequently, the use of larger quantities of higher quality data was substituted for operator insight as the central component in the balancing procedure.

This chapter presents an overview of the development of the rotor balancing literature, with particular emphasis on flexible rotor balancing. Specific reference is made to a number of the more significant contributions and detailed discussions of many of these references are presented in Appendix B. A fairly comprehensive review of the flexible rotor balancing literature prior to 1972 was presented by Rieger [12] and has been invaluable as a guideline and source in the assembly of this literature review. A discussion of the general and rigid-rotor balancing literature is followed by separate discussions of the literature on modal and influence coefficient balancing. Technical discussions of the particular balancing methods are deferred to the next chapter.

## 3.1 General and Rigid Rotor Balancing Literature

Most mechanical engineering handbooks and general references include some mention of rotor balancing. In most cases, this mention is strictly limited to rigid rotor balancing. A number of technical papers concerned with general and rigid rotor balancing have also been published.

One of the earliest general references to include a discussion of rotor balancing was that of Den Hartog [2] in 1934. Most of the emphasis in this discussion was on various methods and machines used for rigid rotor balancing at the time of publication. However, a brief discussion of flexible rotor balancing was also included. A very simple single plane flexible rotor balancing procedure was proposed.

A similar discussion was presented by Timoshenko [3], in 1928, in which several rigid rotor balancing machines of that era were described in considerable detail. The need for special balancing procedures for flexible rotors was mentioned, but no such procedures of a practical nature were proposed.

More recently, similar discussions were presented in both editions of The Shock and Vibration Handbook [13, 14], in 1961 and 1976. While the two editions presented slightly different discussions, the content was basically the same. The primary emphasis was on an updated review of machines and methods for balancing rigid rotors. A brief discussion of flexible rotor balancing was also presented. However, no particular flexible rotor balancing procedures were discussed in any detail.

Discussions of rotor balancing may also be found in some general mechanical engineering references, such as Mark's Handbook for Mechanical Engineers [15]. Such discussions are generally limited to the need for balancing rigid rotors, and a brief description of rigid rotor balancing machines.

In the early 1940s, Kroon published two papers on rotor balancing [16, 17], which were apparently intended as a design guide.

In the first of these papers, Kroon described the theory behind synchronous rotor vibration and the need for balancing of both rigid and flexible rotors. In the second paper, he discussed a number of specific rotor balancing machines and methods. While this discussion was primarily concerned with rigid rotor balancing, a graphical method was described for two plane balancing of flexible rotors. He also presented a brief, practically oriented discussion of field balancing.

A number of other papers concerned with general rotor balancing have been published, including papers by Muster and Flores [18], Jackson [19], and Stadelbauer [20]. Muster and Flores compiled rigid rotor balancing criteria from a variety of sources and compared these criteria with the actual criteria used in American industry at the time (1969). Jackson described a procedure for single plane field balancing of rigid or flexible rotors using an oscilloscope lisajou pattern of the rotor orbit. Stadelbauer presented a review of the history of rigid rotor balancing machines.

## 3.2 Modal Balancing Literature

Modal balancing was developed considerably earlier than influence coefficient balancing. The reasons for this were discussed above.

An early balancing technique, akin to modal balancing, was described by Grobel [21] in 1953. This method was a trial mass procedure which was designed specifically for balancing turbine-generator rotors, making use of known mode shapes for the first three bending modes. Grobel also presented detailed descriptions of several practical problems associated with turbine-generator balancing.

Meldal [22] first developed the orthogonality relations governing modal rotor response, and thus laid the theoretical foundation for modal balancing in 1954. Based on these relations, he outlined the principles of modal balancing (one mode at a time), and described the procedure for balancing a rotor through its first three modes. In this discussion, the measurement of rotor whirl was limited to the bearing locations. However, in general, this is not a necessary limitation.

Beginning in 1957, a series of papers on modal balancing have been published by Bishop and his coworkers, particularly Parkinson. These papers have been concerned with both the theory and application of modal balancing. The flexible rotor balancing method developed in these papers is that most often associated with the term "modal balancing." In the first of these papers [23], Bishop derived a modal series representation of the synchronous whirl of a lightly damped rotor, based on the Jeffcott model analysis. This analysis was extended by Gladwell and Bishop [24] to obtain natural frequencies and mode shapes of an axi-symmetric shaft of nonuniform cross-section on flexible bearings.

The first paper in this series to consider the balancing of flexible rotors was Bishop and Gladwell [25], in which the "modal balancing" method was first proposed. It was shown analytically that "low speed" balancing is entirely inadequate for a flexible rotor. The effects of shaft bow and gravity sag on unbalance response were also considered analytically and found to be, in general, quite small.

A number of papers followed in which the "modal balancing"method was extended or evaluated in light of specific practical problems or conditions which may be encountered in the balancing of flexible rotors. Bishop and Parkinson [26] considered the situation in which the resonant speeds for two, or more, modes are not separated sufficiently to obtain reasonably isolated modes. They discussed the disadvantages of "modal balancing" (as previously outlined) in such a situation, and proposed a modification, adapted from the Kennedy and Pancu [27] method of resonance testing, for such a case.

Lindley and Bishop [28] considered the practical aspects of applying "modal balancing" to large steam turbine rotors in 1963. Actual experience with balancing such rotors is discussed by the authors, as well as by several discussers. Parkinson and Bishop [29] considered the effect of residual vibration after balancing due to the higher, unbalanced modes. A method is proposed by which an averaging technique is used, together with an additional balancing plane, to reduce this residual vibration.

Bishop and Parkinson [30] presented a further discussion of the effect of gravity sag on rotor whirl, with particular emphasis on the second-order whirl found in rotors with dissimilar lateral shaft stiffness (e.g., two-pole generators). The stability of such a rotor was also considered. Bishop and Mahalingam [31] discussed the results of a series of tests with a rotor of this type and extended the discussion of reference [30]. Parkinson [32] then proposed a technique for balancing such rotors, as an extension of "modal balancing."

Parkinson [33] also considered the effect of anisotropic bearing properties (in terms of the non-rotating coordinate system) on the response of a uniform rotor. The effect of the degree of nonsymmetry was discussed and a further extension of "modal balancing" to handle such cases was proposed.

Parkinson [34] summarized the work done by this group in the areas of unbalance response, stability, and balancing of flexible rotors, and presented a complete outline of the "modal balancing" technique. In 1972, Bishop and Parkinson [35] restated the case for "modal balancing" (N plane) and claimed that the additional expense of (N+2) plane balancing is not justified by the resultant insignificant decrease in residual vibration. Discussions by Kellenberger and Federn disputed this point. Fawzy and Bishop [36], and Parkinson [37], discussed the modal interpretation of measured rotor vibration and bearing effects, particularly as related to the application to "modal balancing" (1976).

A number of other researchers have also published in the general area of modal balancing. Federn [38] and Dimentberg [39] published state-of-the-art reviews of modal balancing in the mid-1960s. Moore and Dodd [40] discussed the practical application of modal balancing to turbine-generator rotors in 1964. Actual experimental results were presented with particular emphasis on a rotor which exhibited significant mixed mode behavior. Kushue and Shlyaktin [41] proposed a modal balancing procedure, coupled with a method for analytically predicting mode shapes. As in most modal balancing techniques (Parkinson's method being a notable exception), the requirement for analytically

predicted critical speeds and mode shapes is an integral component of the balancing method. This is generally considered to be a disadvantage, since the typical error associated with such analytical results generally has a significant detrimental effect on the balancing results.

Also in the mid-1960s, Kellenberger [42] proposed a balancing technique similar to the modal balancing techniques discussed above, with one notable exception. He insisted that it is advantageous to balance a flexible rotor initially as a rigid rotor, prior to performing flexible rotor balancing. Then, in order to maintain the rigid rotor balance during flexible rotor balancing it is necessary to use two additional balancing planes, thus the (N+2) plane method mentioned above. In 1972, Kellenberger [43] continued his argument for (N+2) plane balancing (with a discussion by Bishop and Parkinson). A further discussion of the N method and the (N+2) method may be found in Chapter 4. Kellenberger's procedure requires analytical prediction of the rotor mode shapes.

Additional contributions to modal balancing have been made by Church and Plunkett [44], and Hundal and Harker [45]. Neither of these papers presented anything particularly new, although the use of shake-test measurements for the determination of mode shapes in the former paper is interesting. This is, of course, helpful only for a rotor with very well-defined and easily modeled bearing properties.

In 1976, Lund [46] proposed a variant on modal balancing intended to expand the range of applicability and accelerate the procedure of modal balancing. This acceleration is achieved by reducing

the number of trial mass runs to, at most, one per bearing. While analytical predictions are still required, they involve only the free-free mode shapes of the shaft. These can generally be predicted with a somewhat higher reliability than can the bearing support properties. No tests results were presented to verify this technique, though it was stated that a test program was being planned.

Giers [47] evaluated the N and (N+2) plane modal balancing methods, as well as influence coefficient balancing, from the viewpoint of practical application. Of particular interest is a discussion of the practical aspects of data acquisition in rotor balancing.

### 3.3 Influence Coefficient Balancing Literature

Some of the earliest work in rotor balancing was performed by Thearle [48] in 1934. He developed a two-plane balancing method, particularly suited to field balancing, which can be considered to be the forerunner of influence coefficient balancing. Thearle's method was essentially a two-plane, two-sensor, single-speed, exact-point influence coefficient balancing procedure. The calculations required, even for this simple case, were fairly involved. Consequently, no effort was made toward generalizing the influence coefficient procedure, which requires much more complex calculations, before digital computers became generally available. While Thearle made no specific mention of flexible rotors, it appears that his method would have been suitable for some simple, flexible rotors.

A few years after Thearle's paper, Baker [49] presented a slight generalization of Thearle's method which was essentially a

limited, exact point influence coefficient procedure. Baker also suggested some modifications to this method which, while presumably suitable for rigid rotors, are not at all appropriate for flexible rotors. He also discussed the designs of two balancing machines for rigid rotors.

The influence coefficient procedure, as such, was apparently first introduced by El-hadi [50] in a very basic form, some years later (early 1960s). This procedure was proposed in a more formalized and expanded form by Goodman [51] in 1964. This paper is generally considered to represent the introduction of practical influence coefficient balancing. Goodman extended the basic influence coefficient procedure to include least-squares and weighted least-squares solutions. In an interesting discussion to this paper, Heymann [51] described a similar balancing procedure which was in use elsewhere and discussed some of the practical considerations involved in the use of such a procedure.

Lund added some enhancement to the influence coefficient balancing method in a computer program reported by Rieger [52] in 1967. In this reference Rieger also presented the results of an analytical study of the effectiveness of the influence coefficient method which were, in general, very favorable (also reported in [53]). In 1972, Lund and Tonneson [54, 55] reported the details of Lund's influence coefficient procedure along with the results of an experimental investigation which verified its effectiveness. A continuation of this work was reported by Tonneson [56, 57].

At about the same time, LeGrow [58] proposed an alternative balancing method, identical to exact-point influence coefficient balancing except that the influence coefficients are determined analytically. This produces the obvious advantage of eliminating the trial mass runs. However, analytical predictions of this type are prone to significant error which can have a considerable detrimental effect on the results (and efficiency) of the balancing procedure. Also, LeGrow's claim that an exact point solution is always preferable to a least-squares solution is clearly open to dispute.

Tang and Trumpler [59], and Yanabe and Tamara [60] proposed different balancing procedures which are both vaguely related to influence coefficient balancing and are both strongly associated with analytical rotor response prediction procedures. Tang and Trumpler proposed a method whereby "disk sensitivities" are predicted analytically and the balancing is accomplished by adjusting the angular orientation of the disks. Yanabe and Tamara used an analytical procedure for determining a matrix of "stiffness coefficients" which are subsequently used in balancing calculations. Apparently, neither of these methods has seen any further development.

Over a period of about ten years (1970s), a large number of experimental investigations were performed by Tessarzik and his coworkers at MTI to evaluate Lund's influence coefficient procedure and computer program. These test programs, as well as several analytical studies, were intended to determine the effectiveness and practicality of influence coefficient balancing for a variety of different rotor types, sizes and operating conditions. Based on the

experience gained in these studies, a number of modifications and extensions were made to Lund's program.

In the first of these tests, Tessarzik [61] used the exact point-speed influence coefficient procedure (total number of readings equal to number of planes) to balance a three-mass rotor through two rigid and one flexural critical speed. The results of these tests, along with a discussion of the practical aspects of applying this balancing method, were presented by Tessarzik, Badgley and Anderson [62]. A second series of tests was performed by Tessarzik [63], using the same test rig, to evaluate the least-squares influence coefficient balancing method. These tests were expanded to include other rotor configurations (various disk masses) and more difficult initial unbalance conditions. The results of these tests were compared with those from the exact point tests to evaluate the relative merits of the two procedures [64]. In general, the least-squares method was found to be superior. Tessarzik [65] subsequently performed a third series of tests with this test rig. The rotor was modified to provide four flexural critical speeds in the operating speed range. However, two of these modes were heavily damped and did not require balancing. These tests were conducted for a number of test rig operating and unbalance conditions, including both rigid and flexible bearing pedestals. In the case of the flexible pedestals, the supports were intentionally designed to be anisotropic (softer horizontal than vertical support), resulting in rather extreme elliptical orbits [66]. In all cases, the balancing procedure was quite successful.

This same test rig was later modified by Smalley, Tessarzik and Badgley [67] to provide substantial nonsymmetry of the lateral stiffness of the rotor. While the principal purpose of these tests was apparently to investigate the stability of such a rotor, the balancing of the nonsymmetric rotor proved to be quite interesting. The least squares influence coefficient balancing procedure verified earlier by Tessarzik [65] was used very successfully for this rotor [68]. This balancing procedure was incorporated into a minicomputer-based automatic data acquisition and computation system, as described in an appendix to reference [68]. The software used in this balancing system, which is an adaptation of Lund's original program, was documented by Darlow and Fanuele [69]. In another appendix to reference [68], Lund provided an interesting discussion on the optimization of balancing plane and sensor locations for influence coefficient balancing.

Much of the balancing work performed at MTI has concentrated on particular rotor types which have tendencies to be very flexible. Two of these rotor types are gas turbine engine power turbine shafts and high-speed power transmission shafts. The investigations into the application of influence coefficient balancing to gas turbine engines began with an analytical feasiblity study by Rieger and Badgley [70]. This work was subsequently extended and generalized by Badgley and Rieger [71]. Badgley [72, 73] discussed some of the practical implications of balancing gas turbine rotors. Badgley et al. [74] and Badgley [75, 76] also presented similar discussions for a more general

class of flexible rotors. Cundiff, Badgley and Reddecliff described a test rig designed to simulate a gas turbine engine power turbine shaft [77], and the balancing of this simulator through two flexural critical speeds using influence coefficient balancing [78]. A related application of influence coefficient balancing involved the successful simultaneous balancing of a pair of marine steam turbines by Tessarzik, Darlow and Badgley [79].

A similar investigation of high-speed power transmission shafts began with an analytical study by Badgley and Tessarzik [80]. In this paper, the feasibility of designing, manufacturing and balancing a supercritical power transmission shaft was demonstrated, at least theoretically. This followed earlier work on high-speed power transmission shafts by Baier and Mack [81]. The work of reference [80] was later extended and generalized by Badgley, Smalley and Fleming [82]. A scale model test rig (approximately 30 feet long) of such a power transmission shaft was subsequently designed and built, as reported by Darlow and Smalley [83]. Several series of balancing tests were conducted with this test rig (some of which are described in detail in Chapter 7). These tests included influence coefficient balancing of an unloaded shaft (in terms of torque), as described by Darlow and Smalley [84]; and similar tests with a shaft operating under a substantial torque load, as reported by Darlow, Smalley and Fleming [85]. These tests involved influence coefficient balancing of a test shaft through three active (four total) flexural critical speeds.

Another area of investigation at MTI involved the use of high-power lasers for removing metal for rotor balancing. Applying lasers in rotor balancing in this way serves the dual purpose of fully automating the balancing procedure and permitting the application of compensating unbalance without stopping the rotor. The second of these points is particularly advantageous for rotors that require excessive time for startup or shutdown (e.g., large steam turbines). The use of a laser for single plane balancing of simple rigid rotors has been demonstrated by Popick and Roberts [86], Dobson [87], and Damon [88]. A study was subsequently conducted by Tessarzik and Fleming [89] to evaluate the feasibility of incorporating laser metal removal into the multiplane influence coefficient balancing process. The results of this study were quite encouraging. Consequently, the test rig of reference [65] was modified to include a microprocessor-controlled laser metal removal system, as reported by DeMuth, Fleming and Rio [90]. This system was successfully used for fully automatic two-plane balancing.

The last aspect of the balancing work conducted at MTI involved the practical application of balancing procedures. In general, two areas were considered: field balancing and permanent installation balancing. The complex calculations required for influence coefficient balancing are not, in general, compatible with field balancing. However, the advent of the telephone access time-sharing computer has permitted the marriage of influence coefficient balancing with field balancing (where the vibration data are entered by the user with a portable computer terminal). For this reason, a group of interactive influence coefficient balancing computer

programs, based on Lund's program, have been developed and documented by Darlow and Badgley [91, 92, 93, 94]. The first of these programs [91] is essentially a direct adaptation of Lund's program for interactive use. The second program [92] uses known influence coefficients (either from previous balancing tests or from other similar rotors) and thus does not require trial mass runs. The last two programs [93, 94] are modifications of the first two of these programs to include several features which were found to be of practical use.

The development of the minicomputer has made possible very sophisticated, permanent installation, influence coefficient balancing systems. A number of such systems have been developed for particular applications, such as those documented by Fanuele and Darlow [95], and Fanuele [96].

Analytical and experimental investigations involving influence coefficient balancing have been conducted by a great many other researchers during the past decade. While the majority of these studies have not involved the addition of anything new to the concept or practice of influence coefficient balancing, a few of them have involved some interesting and innovative proposals.

Little and Pilkey [97] proposed a modified influence coefficient method based on a linear programming approach, in 1975. This method requires the number of balancing planes to exceed the number of vibration readings. Thus, an infinite number of solutions are possible. The optimum solution is chosen such that an objective function is minimized. This objective function is generally taken to be the residual vibration at a speed (or condition) at which

observations cannot be made (using analytically determined influence coefficients). The ordinary influence coefficient equations are then treated as constraints. An additional constraint may be added to require the sizes of the correction masses to fall within an acceptable range, thus preventing the calculation of excessively large, or small, correction masses. This balancing method was described in more detail by Little [98], and verified with some interesting analytical examples. However, no experimental results were reported. A similar balancing method was described by Gleizer [99] for use in gas turbine engine flexible rotor balancing.

Nicholas, Gunter and Allaire studied the unbalance response [100] and balancing [101] of a single mass flexible rotor with residual shaft bow. They determined that the optimum results were obtained by reducing the total rotor amplitude (including shaft bow) to zero at the critical speed, and to levels less than or equal to the bow at other speeds. However, this requires prior knowledge of the critical speed, which is not often known with sufficient accuracy. It was shown that an acceptable balance could be achieved by balancing the rotor to the residual bow amplitude at a speed (or speeds) in the general vicinity of the critical speed. Balancing the total rotor amplitude to zero at a speed away from the critical speed was shown by the authors to give totally unacceptable results.

In 1976, Larsson [102] presented a statistical procedure for calculating influence coefficients from multiple sets of trial mass data. He used a linear regression analysis to determine the influence coefficients that result in a "best fit" to the experimental data.

The algorithm was constructed in such a way that data from runs with groups of (trial or correction) masses may also be used. In addition to the influence coefficients, Larsson's algorithm also produces a coefficient of multiple determination and a coefficient of reliability. The former of these coefficients provides a measure of the correlation between the influence coefficients and the trial mass data, while the latter provides a measure of the reliability of the calculated correction masses. Larsson discussed the results of tests using two productions runs of two types of rotors (31 of one rotor type and 2 of the other) in which the influence coefficients were calculated, and continually improved, through the use of his algorithm. The use of these influence coefficients aided substantially in the balancing of these rotors and permitted the use of fewer trial mass runs (or even elimination of the trial mass runs altogether) toward the end of a production run.

Drechsler [103] has proposed a similar procedure for applying the tools of statistical analysis to the calculation of influence coefficients. However, his discussion was concerned primarily with feasibility and general approach, and lacked the detail and development of Larsson's presentation, Drechsler did not present any experimental verification.

In an essentially unrelated paper, Drechsler [104] proposed a balancing method which is, to a certain extent, a combination of modal and influence coefficient balancing. He simply applied the standard influence coefficient equations and added a weighting matrix designed to account for the analytically predicted mode shapes of the rotor.

Individual trial masses are used throughout this balancing procedure. Thus, while this method is not prone to the potential numerical problems of influence coefficient balancing, no attempt is made to deal with the inherent difficulty of obtaining reasonable trial mass data at speeds above one or more of the lowest flexural critical speeds for a lightly damped system. Also, the use of analytically determined mode shapes is a distinct disadvantage borrowed from modal balancing. Drechsler presented a numerical example to verify his balancing method. However, no experimental verification was provided.

In 1977, Bigret, Curami, Frigeri and Macchi [105] described a modified influence coefficient balancing method adapted for use with an in-field computer. This balancing method is the standard weighted least-squares influence coefficient method modified to use a generalized weighting matrix (as described in Chapter 4) to permit relative importance weighting of particular speeds or sensors. The in-field computer system used relied on manual input of vibrations data by means of a teletype. The use of such a system with computerized data acquisition would result in a tremendous improvement in usability and accuracy (as described in Chapter 6). This balancing system was experimentally verified, with good success, as described in the paper.

Fujisawa, Siobato, Sato, Imai and Shoyama [106] reported good results from an experimental investigation in which the least-squares influence coefficient method was used for balancing a multispan, flexible rotor, in 1979. Although there was nothing innovative about

their balancing method, this does appear to be the first publication of actual balancing results for multispan rotors using an influence coefficient balancing procedure. Since such rotors are of considerable practical importance (e.g., modern steam turbine-generator sets), this paper is worthy of note.

# CHAPTER 4
# CURRENT ROTOR BALANCING METHODS AND THEORY

The mass balancing of rotating machines (referred to more briefly as rotor balancing) has been an important consideration in machinery design and manufacture for more than forty years. Numerous methods and machines have been devised for rotor balancing and a substantial body of related literature has appeared during this period of time. A substantial portion of this literature was reviewed in the previous chapter and detailed reviews of a number of the more pertinent publications are presented in Appendix B.

Rotor balancing methods may be separated into two categories: rigid rotor balancing and flexible rotor balancing. These generic titles refer to whether or not rotor flexibility is taken into account. That is, flexible rotor balancing procedures take into account rotor flexibility, while rigid rotor balancing procedures do not. Many flexible rotor balancing methods are suitable for balancing rigid rotors. However, rigid rotor balancing methods are, in general, not effective for balancing flexible rotors. Unfortunately, in practice, the use of rigid rotor balancing methods with flexible rotors is not unusual. The result is, at best, an extremely inefficient and costly balancing process and, at worst, a presumably well-balanced rotor which, at its operating speed, is actually very poorly balanced and potentially destructive. Specific rigid and flexible rotor balancing

methods are discussed in considerable detail in the following two sections.

Field balancing is a term which appears repeatedly in the literature and deserves some explanation here. This term refers to a situation in which a balancing method is applied and is not a balancing procedure or classification, as such. Field balancing refers to some type of on-site balancing, generally using portable instrumentation. The rotor being field balanced may be either rigid or flexible. However, the procedures most commonly used for field balancing utilize only a single balancing plane [19] or multiple balancing planes in a trial-and-error fashion (such as the polar plot methods described below). In either case, balancing of flexible rotors is, at best, inefficient and more likely, ineffective. Some efforts toward development and promotion of systematic flexible rotor balancing procedures suitable for field balancing are currently underway [64, 72, 91-94], primarily in the field of influence coefficient balancing. However, general acceptance of these systematic procedures has not occurred, particularly in field balancing, primarily due to several inherent disadvantages of influence coefficient balancing which are discussed in considerable detail in a later section. On the other hand, the Unified Balancing Approach, which is the principal subject of this presentation, is easily applied to field balancing for rigid or flexible rotors, and is not subject to the deficiencies of influence coefficient balancing.

## 4.1 Rigid Rotor Balancing

The classification of rigid rotor balancing encompasses a number of balancing procedures whose common feature is the assumption that the rotor being balanced does not elastically deform at any speed up to its maximum design speed. If the rotor being balanced does not satisfy this assumption, rigid rotor balancing is not adequate and flexible rotor balancing, as described in the next section, must be used.

Rigid rotor balancing is divided into two distinct classifications commonly referred to as "static" balancing and "dynamic" balancing. Static balancing refers to single-plane balancing, which is appropriate for rigid rotors in which the mass is concentrated in a single narrow disk. Thus, the removal of any mass eccentricity in the disk effectively balances the rotor as a whole. The single-degree-of-freedom rotor model illustrated in Figure 2-12 is a classic representation of such a rotor. Examples of actual rotors appropriate for static balancing include gyroscopes and bull gears on short, rigid shafts. In general, many types of single disk straddle mounted rotors with short bearing spans may be effectively balanced by static balancing.

Dynamic balancing refers to two-plane balancing. This is somewhat of a misnomer in that it implies that the balancing takes into account all of the dynamics of the rotor (including rotor deformation). This, of course, is not the case, as dynamic balancing is limited in application to rigid rotors. However, as discussed in Chapter 2, any rigid rotor can be completely balanced in two planes,

regardless of whether it is rigidly or flexibly mounted. This conclusion is based on the general rigid rotor model illustrated in Figure 2-17 and the accompanying analysis.

In particular, any general unbalance distribution may be represented as a linear combination of an a and a $\theta_a$ and some arbitrary unbalance distribution which is completely self-cancelling. A self-cancelling unbalance distribution is one which results in no forces or moments being transmitted to the bearings. That is, the resulting unbalance forces satisfy

$$\int_0^{\ell} u_c(z)dz = 0 \tag{4-1}$$

$$\int_0^{\ell} zu_c(z)dz = 0 \tag{4-2}$$

where $u_c(z)$ represents the self-cancelling unbalance distribution; z is measured in the axial direction; and $\ell$ is the length of the rotor. An example of a general unbalance distribution and its components is given in Figure 4-1. Although the unbalance distribution components in this figure are shown to exist in a single diametral plane, this is done for clarity only. In general, these components are found to exist in diametral planes which are rotated with respect to one another.

The a component of unbalance may be canceled with a single correction mass and a pair of correction masses are required to cancel the $\theta_a$ component of unbalance. However, no correction is required for

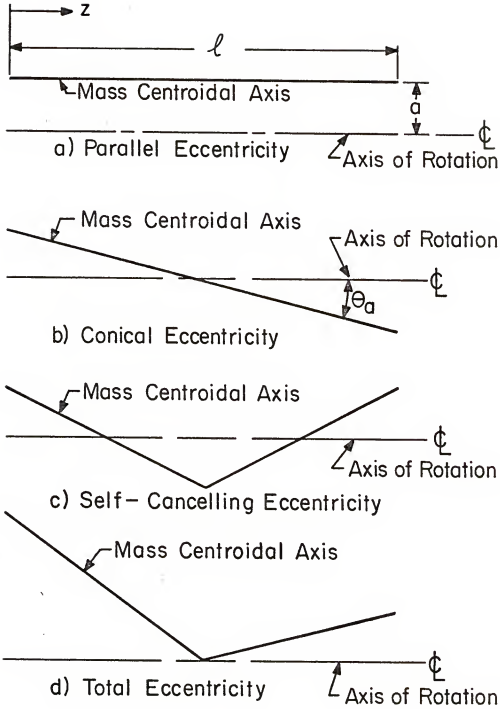



Figure 4-1 Distribution of mass centroidal axis eccentricity along the axis of rotation of a rigid rotor and the effective components in terms of rigid rotor response

the self-cancelling component. Since the axial location of the correction mass for the a component is arbitrary, this mass may be located in the same diametral plane as one of the correction masses for the $\theta_a$ component. Since two correction masses located in the same diametral plane can always be replaced by the single mass which represents their vector sum (in rotating coordinates), it becomes apparent that a rigid rotor can be always be balanced in, at most, two planes. Thus, dynamic balancing is always sufficient for a rigid rotor. Note that if the correction for a is not located in the same transverse plane as the center of gravity, $\theta_a$ will be modified.

The differences among the various existing machines and methods for rigid rotor balancing are more closely related to implementation than to theory. Specific machines and methods for both static and dynamic balancing are discussed in some detail.

One of the oldest and most commonly used methods for static balancing involves the use of horizontal knife-edges. The rotor is set on the knife-edges and is permitted to rotate until the heavy side faces down. Some mass is then attached to the light side (the top) of the rotor and the process is repeated until the rotor has no preferential orientation. A static balancing machine which uses a similar procedure is the bubble-type automobile tire balancer. One of the primary drawbacks of this machine is that if the center of rotation of the axle relative to the wheel is not exactly the same as the pivot point on the tire balancer (which is invariably the case, to some extent), the balance condition of the tire, wheel and axle assembly may not be very good.

A somewhat more systematic method of static balancing (or single-plane balancing, in general), which is in particularly common use in field balancing, is referred to as the "polar plot" method. (Obviously, the knife-edge procedure is not suited to field balancing.) In the "polar plot" method, rotor vibration is first measured, generally at a low speed, for the initial unbalanced condition (referred to as "uncorrected rotor data"). Then one or more trial mass runs are conducted where a trial mass of known size is installed in the rotor at a predetermined angular location and rotor vibration is measured at the same speed as for the uncorrected rotor data. Using this rotor vibration data and fairly simple polar plotting techniques, the appropriate unbalance compensation (or correction) masses are calculated and installed in the rotor. If the rotor vibration is not sufficiently reduced (which is not uncommon) the entire procedure is repeated to refine the correction masses.

If vibration data can be measured in terms of amplitude and phase angle (relative to an angular reference fixed in the rotor) only a single trial mass run is required for a correction mass calculation. However, if only vibration amplitude is available, then a minimum of two trial mass runs are required. In general, three trial mass runs are used in this situation since the use of only two trial mass runs permits specification of the diametral plane for the correction mass, but not the direction (there are two possible choices which are 180 degrees out of phase). The use of three trial mass runs allows the complete specification of the correction mass. The "polar plot" method is, in essence, a very simplified, single-plane influence coefficient procedure.

In general, dynamic balancing machines and methods are perfectly adequate for static balancing, although the converse is certainly not true. Many ingenious methods were developed for dynamic balancing prior to the advent of modern vibration instrumentation. Most of these methods are no longer used and a discussion of them is beyond the scope of this study. However, detailed descriptions of some of these methods may be found in references [2, 3, 17, 20 and 49]. Of primary interest here are the dynamic balancing machines which are currently in most common use. These machines generally fall into two classifications, commonly referred to as soft-bearing machines and hard-bearing machines, which very aptly describe their differences.

Soft-bearing balancing machines have rotor supports which are designed to be very soft in the horizontal direction, such that the rigid rotor critical speeds are well below the balancing speed. Thus, the rotor support vibrations are sufficiently large to permit high-resolution displacement measurement. The principal advantage of soft-bearing machines is the excellent sensitivity, particularly for light rotors for which sensitivity is often a problem with hard-bearing machines.

With the soft-bearing machines, either trial mass runs are required to "calibrate" the rotor, or else a less systematic trial-and-error procedure must be used. With the development of the hard-bearing machines, the popularity of the soft-bearing machines has decreased, particularly for production run balancing. This is due largely to the inconvenience of rotor "calibration," the difficulty of handling massive rotors, and the need for relatively skilled

operators for soft-bearing machines. Currently, these machines are used more commonly for high-precision, low-volume balancing requirements and for balancing of very light rotors.

Hard-bearing balancing machines, introduced more recently than soft-bearing machines, were made possible by the development of sensitive and accurate force measurement transducers (or force gages). In hard-bearing machines, the rotor supports are structurally rigid and are each mounted on one or more force gages. Since the force gages are also very stiff, the balancing speed is far below all rotor critical speeds. Consequently, the rotor vibration (whose displacement amplitudes are very small) and bearing forces (which are measured by the force gages) attain maximum values essentially directly in line (angularly) wih the rotor unbalance. Also, the bearing forces may be related directly to the mass eccentricity of the rotor, if the rotor mass distribution is known, or to the equivalent discrete unbalances, if the axial locations and radii of the unbalances are known. In general, the axial locations of the unbalances are chosen and the corresponding radii may be easily measured. Thus, the equivalent discrete unbalances can be calculated directly from the bearing forces and the appropriate discrete correction masses can be applied to the rotor.

As discussed above, no more than two balancing planes are required for complete balancing of a rigid rotor. Figure 4-2 illustrates a typical rigid rotor-balancing setup. For a constant speed of operation, the synchronous components of the bearing forces may then be treated as quasi-static forces which are represented by

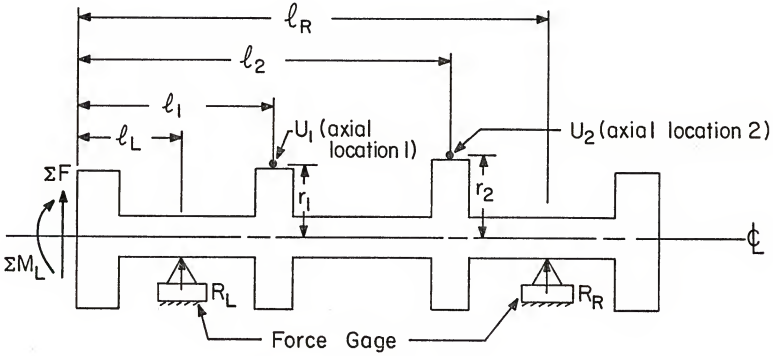



Figure 4-3 Typical rigid rotor balancing setup

complex numbers, as discussed in Chapter 2. Then, if the moment carrying capability of the rotor supports is assumed to be negligible (which is generally one of the design objectives), the rotor may be treated as being statically determined. Thus, using simple statics (but with complex valued forces)

$$\Sigma F = R_L + U_1 r_1 \omega^2 + U_2 r_2 \omega^2 + R_R = 0 \tag{4-3}$$

$$(\Sigma M)_L = R_L \ell_L + U_1 r_1 \omega^2 \ell_1 + U_2 r_2 \omega^2 \ell_2 + R_R \ell_R = 0 \tag{4-4}$$

where

$\Sigma F$ represents the sum of the forces on the rotor

$(\Sigma M)_L$ represents the sum of the moments about the left end of the rotor

$R_L$, $R_R$ are the bearing reactions at the left and right rotor supports, respectively

$U_1$, $U_2$ are the unknown equivalent discrete unbalances at axial locations 1 and 2 in Figure 4-2

$r_1$, $r_2$ are the corresponding radii for application of the correction masses

$\ell_1$, $\ell_2$ are the axial distances from the left end of the rotor to locations 1 and 2 in Figure 4-2

$\ell_L$, $\ell_R$ are the axial distances from the left end of the rotor to the left and right rotor supports, respectively, and

$\omega$ is the speed of rotation in radians per second

Note that the bearing reactions are 180 degrees out of phase with the bearing forces. Equations (4-3) and (4-4) may be solved for $U_1$ and $U_2$ to give

$$U_1 = [R_R(\ell_R - \ell_2) - R_L(\ell_2 - \ell_L)]/[r_1\omega^2(\ell_2 - \ell_1)] \qquad (4\text{-}5)$$

$$U_2 = [R_L(\ell_1 - \ell_L) - R_R(\ell_R - \ell_1)]/[r_2\omega^2(\ell_2 - \ell_1)] \qquad (4\text{-}6)$$

where the result is complex valued. The calculation of Equations (4-5) and (4-6) may be accomplished with analog electronics (as was done originally) or with digital components (as in some of the more advanced balancing machines). The ability to calculate $U_1$ and $U_2$ simultaneously is often referred to as "plane separation."

If the correction masses are to be applied by removal of mass from the rotor (e.g., by grinding the rotor), $U_1$ and $U_2$ define the amount of mass to be removed from the rotor and the appropriate angular orientations (i.e., the unbalances $U_1$ and $U_2$ are being removed). If, however, the balance corrections are to be made by the addition of mass to the rotor (e.g., by placing set screws in pre-threaded holes), the angular orientations of the corrections must be rotated 180 degrees (opposite side of the rotor) from that defined by $U_1$ and $U_2$ (i.e., the unbalances $U_1$ and $U_2$ are being compensated).

The popularity of hard-bearing balancing machines is currently increasing for generally the same reasons that the popularity of soft-bearing machines is declining. The convenience of one-shot balancing and suitability for production operation are two attractive features of hard-bearing machines. In addition, these machines can be used by relatively unskilled operators and are capable of handling a wide range of rotor sizes.

Typically, the balancing speed for hard-bearing machines is between 600 and 1800 rpm. Generally, the high speeds are used for lighter rotors or where tight balancing tolerances are encountered. The bearing forces (and thus the sensitivity to unbalance) is proportional to the square of the speed of rotation. Therefore, when more sensitivity is required higher balancing speeds can be used. The balancing speed is, however, limited by the flexibility of the rotor and supports in that it must remain well below the lowest rotor critical speed. Sometimes, particularly in the case of very light rotors, it is not possible to attain sufficient sensitivity at allowable balancing speeds. In this case, it is necessary to go to a soft-bearing balancing machine.

There are certain potential problems associated with the use of rigid rotor balancing machines. These problems are generally a result of improper or inappropriate use of rigid rotor machines and can be avoided if the supervisory engineer is aware of them. First of all, it is essential that rigid rotor machines not be used for balancing rotors which are, in fact, flexible. Second, rotors should have the same centers of rotation on the balancing machine as they do

in operation. For example, if a rotor is to be supported by rolling-element bearings, it should, if possible be balanced mounted in these same bearings. If the same center of rotation is not used, substantial unbalance can be introduced which may have a detrimental effect on bearing and rotor life. This is particularly true in cases where rotor disks are balanced individually on arbors rather than on their shafts. What often happens in this case is that the disk is balanced on an arbor that has very little eccentricity and is then mounted on a shaft with considerably more eccentricity. The result is a large unbalance equal to the mass of the disk times the eccentricity of the shaft. In cases where it is not possible to duplicate a rotor's operating centers of rotation on a rigid rotor balancing machine, it may be necessary to balance the rotor "in-situ" (completely assembled, in place) or "quasi-in-situ" (in a specially constructed balancing rig which simulates the actual machine).

A third source of potential problems with rigid rotor balancing machines occurs when a rotor stack-up must be disassembled after balancing in order to be installed. Considerable rotor unbalance is often introduced in this way, even if care is taken to reassemble the rotor components without changing their angular orientations. This problem can occur with quasi-in-situ balancing as well as when using rigid rotor balancing machines. One of the principal limitations of these balancing machines is that it must be possible to bring the rotor to the balancing machine. In many cases, particularly with very large rotors, this simply is not possible.

## 4.2 Flexible Rotor Balancing

A flexible rotor is defined as one which exhibits significant lateral deformation at, or below, its maximum design speed. A survey of the literature on flexible rotor balancing is presented in Chapter 3. The theoretical development of various flexible rotor balancing procedures is presented in this section.

The use of rigid rotor balancing procedures for flexible rotors can be a very dangerous practice. Such practices are generally ineffective, and, at the least, inefficient. It is not unusual for the use of rigid rotor balancing to actually aggravate the unbalance condition of a flexible rotor at its operating speed. For example, consider the flexible rotor in Figure 4-3. This is a uniform rotor of length $\ell$ and radius r, supported at its ends by relatively rigid bearings. This rotor has a single discrete unbalance mass U located on the surface of the shaft at an axial distance 0.1 $\ell$ from the left end of the rotor. The two correction mass planes (for dynamic balancing) are located axial distances 0.4 $\ell$ and 0.6 $\ell$ from the left end of the rotor. The mode shapes for the first two critical speeds are illustrated in Figure 4-4.

If this rotor is balanced as a rigid rotor at rotational speed $\omega$ (dynamic balancing) by measuring the bearing reaction forces (which are 0.9 $\omega^2 rU$ and 0.1 $\omega^2 rU$ for $R_L$ and $R_R$, respectively) the required correction masses $U_1$ and $U_2$ (also located on the surface of the shaft) from Equations (4-5) and (4-6), are -2.5 U and 1.5 U, respectively. The amplitude of vibration (by any means of measurement) at, or near, a critical speed is proportional to

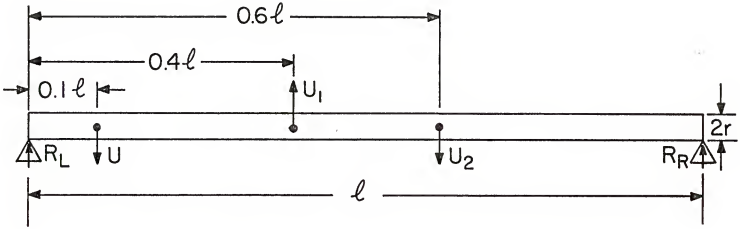



Figure 4-3 Uniform flexible rotor on rigid bearings

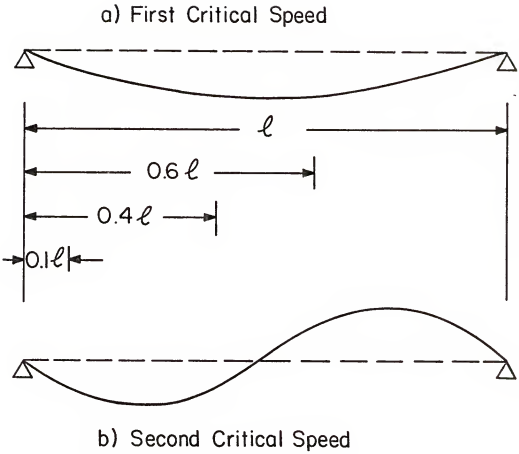



Figure 4-4 Mode shapes for uniform flexible rotor on rigid bearings

$$\sum_{i=1}^{n} U_i r_i x_i \tag{4-7}$$

where the $U_i$ are the n discrete unbalance masses (including the correction masses) and $r_i$ and $x_i$ are the corresponding radii and normalized amplitudes of vibration, respectively. Thus, if without rigid rotor balancing, the vibration amplitudes at the first two critical speeds are proportional to 0.31 rU and 0.59 rU, respectively (where rU is the original unbalance mass moment), then after rigid rotor balancing, the corresponding vibration amplitudes will become proportional to 0.64 rU and 1.76 rU. Thus, for this example, the rigid rotor balancing procedure actually has a detrimental effect on the rotor vibration levels of the flexible rotor. Some flexible rotor balancing procedures require prior rigid rotor balancing. However, in general, rigid rotor balancing alone is not adequate for a flexible rotor.

A flexible rotor must, in general, be balanced in the vicinity of all critical speeds below the maximum design speed of the rotor. It is also often necessary to balance a flexible rotor at its operating speed. Occasionally balancing at an overspeed condition is also required. A typical situation where overspeed balancing is desirable is when a mode for a critical speed above the operating speed range has a significant response at the operating speed (requiring balancing), and the added balance data sensitivity afforded by balancing at overspeed provides for a more precisely balanced rotor. The ultimate aim of any flexible rotor balancing procedure is to produce acceptable vibration levels (bearing reaction forces or displacement amplitudes)

at all balancing speeds, not merely at the running speeds. This is because even for rotors which operate at a constant speed, it must be possible to safely accelerate to running speed and decelerate to shutdown, going through critical speeds.

The vibration data used for balancing of flexible rotors may come from a variety of sources. One commonly used source of vibration data is dynamic bearing reactions. These bearing reactions may be measured in the form of forces (using force gages), displacements (using displacement sensors) or, in the case of film bearings, oil pressure (using pressure sensors). The bearing reactions may also be measured indirectly in the form of bearing pedestal vibrations. The most commonly used device for measuring pedestal vibrations is the velocity pickup (although accelerometers and displacement sensors are also used). The use of pedestal vibration data is most frequently associated with field balancing.

The measurement of dynamic rotor deflection (using displacement sensors) is another important source of balancing data. As rotors have become increasingly flexible, the maintenance of rotor/stator clearance has become a real problem and the use of rotor deflections for balancing data has gained substantial popularity and has, in some cases, become essential. In particular, for extremely flexible rotors mounted on relatively rigid bearings, bearing reaction data may not provide sufficient sensitivity to achieve an acceptable level of balance.

The principal drawback to the use of displacement sensors along a rotor has historically been the general inaccessibility of the rotors

in fully assembled rotating machinery (with the possible exception of the ends of the rotors). However, as machinery manufacturers have begun to recognize the need for rotor vibration monitoring (not only for balancing, but for machinery maintenance and diagnostics, in general), they have started installing displacement sensors and other transducers as integral components of their machines. (The cost of these transducers is negligible when compared to the overall cost of most commercial machinery.) However, a great deal of manufacturer resistance to properly instrumenting rotating machinery remains to be overcome.

When using displacement sensors for measuring rotor deflections, the proper choice of the axial locations of the sensors is very important. If the sensors are all located at, or near, rotor deflection nodes (points of zero lateral displacement), the rotor may appear to be very well balanced when, in fact, it is not. Such a situation is potentially disastrous. Thus, in general, proper selection and location of vibration sensors is critical to successful and efficient balancing of flexible rotors.

Current flexible rotor balancing procedures may be generally separated into two classifications: modal balancing and influence coefficient balancing. Modal balancing procedures are characterized by their use of the modal nature of rotor response. In general, modal balancing procedures are highly developed, but are largely intuitive in nature. By contrast, influence coefficient balancing procedures have been developed more recently (in their present form), but are

entirely empirical. These procedures are based on the generally reasonable assumption of linear superposition of rotor response. Several other balancing procedures are presented in the literature which are essentially specializations of either modal or influence coefficient balancing. The Unified Balancing Approach (UBA), which is introduced in this study, combines the advantages of both modal and influence coefficient balancing while eliminating their drawbacks by taking account of modal effects in a test-supported procedure. The UBA is discussed in detail in Chapter 6. Detailed discussions of modal and influence coefficient balancing are presented below.

### 4.2.1 Modal Balancing

Neglecting the effect of system damping and lateral non-symmetry (inherent assumptions of modal balancing) the principal modes of an axially uniform rotor system are found to satisfy the orthogonality condition

$$(\omega_j^2 - \omega_k^2) \int_0^{\ell} x_j(z) x_k(z) dz = 0 \tag{4-8}$$

where $\omega_j$ and $\omega_k$ are undamped system resonances (critical speeds) for the $j^{th}$ and $k^{th}$ rotor whirl modes, respectively, and $x_j(z)$ and $x_k(z)$ are the corresponding planar (though not necessarily coplanar) modal amplitudes [12]. The Z-direction is along the rotor axis and the overall length of the rotor is $\ell$. For j not equal to k, Equation (4-8) reduces to

$$\int_0^{\ell} x_j(z) x_k(z) dz = 0; \quad j \neq k \tag{4-9}$$

For a more general rotor system with axial nonuniformity, Equations (4-8) and (4-9) may be generalized in the form

$$\int_0^{\ell} A\rho(z)\phi_j(z)\phi_k(z)dz = \begin{cases} 0 & (j \neq k) \\ Q & (j = k) \end{cases} \tag{4-10}$$

where the mode shapes $x_j(z)$ and $x_k(z)$ are now represented by the characteristic functions $\phi_j(z)$ and $\phi_k(z)$, respectively [35]. The term $A\rho(z)$ is the mass per unit length (cross-sectional area times density) of the rotor and Q is the normalizing factor for the characteristic functions. That is, the characteristic functions are normalized such that Q, which has dimensions of mass, is constant regardless of the values of j and k (so long as j = k).

Using the property of orthogonality, any fairly well-behaved function f(z) (such as mass eccentricity and rotor deflection), can be expressed as an infinite series of the characteristic functions (since there are an infinite number of characteristic functions) by

$$f(z) = \sum_{j=1}^{\infty} f_j\phi_j(z) \tag{4-11}$$

where the coefficients $f_j$ are given by

$$f_j = \frac{1}{Q}\int_0^{\ell} A\rho(z)f(z)\phi_j(z)dz \tag{4-12}$$

The actual rotor deflection x(z) may be represented by an equation analogous to Equation (4-11), where the coefficients $x_j$ (in $x(z) = \sum_j x_j\phi_j(z)$) are functions of rotational speed. In general, as the

rotational speed approaches the $j^{th}$ resonance, $x_j$ becomes substantially larger than the other coefficients. Thus, $x(z)$ becomes nearly equal to $x_j\phi_j(z)$. This is referred to as modal response.

As an example, consider a uniform rotor of length $\ell$ supported at its ends by radially rigid (but angularly free) bearings. In the absence of damping and gyroscopic effects, the response of this rotor is identical to the lateral vibration of a nonrotating beam with the same dimensions. Neglecting shear deformation (which is negligible for a long slender beam), the moment, M, applied to a differential section of the beam (see Figure 4-5) is related to the curvature of the beam by

$$M = EI \frac{\partial^2 x}{\partial z^2} \tag{4-13}$$

for small deformations. The modulus of elasticity and cross-sectional moments of inertia are represented by E and I, respectively, while z is the coordinate measured along the rotor axis and x is the lateral deflection. The differentials in Equation (4-13) are partial since x is also a function of time.

For a vibrating beam, neglecting rotatory inertia, the change in shear across the differential beam element is related to the deflection by

$$\frac{\partial V}{\partial z} = -\rho A \frac{\partial^2 x}{\partial t^2} \tag{4-14}$$

where V is the shear, $\rho$ is the density of the beam material, A is the cross-sectional area of the beam, and t is time. Equations (4-13) and (4-14) may be combined, together with the relation

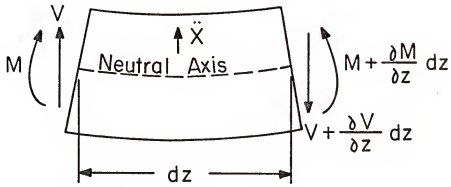



Figure 4-5 Differential element of a beam in bending

$$V = \frac{\partial M}{\partial z} \tag{4-15}$$

to give the governing equation

$$\frac{\partial^2}{\partial z^2}\left(EI\,\frac{\partial^2 x}{\partial z^2}\right) = -\rho A\,\frac{\partial^2 x}{\partial t^2} \tag{4-16}$$

For a uniform beam, E, I, $\rho$ and A are constants and Equation (4-16) may be reduced to

$$\frac{\partial^4 x}{\partial z^4} = -\frac{\rho A}{EI}\,\frac{\partial^2 x}{\partial t^2} = -\kappa^2\,\frac{\partial^2 x}{\partial t^2} \tag{4-17}$$

where $\kappa^2$ is a positive real value (in this case a constant) which represents the ratio of the mass per unit length of the beam to its flexural rigidity.

Equation (4-17) may be solved by separation of variables where x is taken to be

$$x(z,t) = Z(z)T(t) \tag{4-18}$$

where Z has the same dimensions as x (length) and T is dimensionless. Substituting Equation (4-18) into Equation (4-17) and rearranging gives

$$\frac{1}{Z}\,\frac{d^4 Z}{dz^4} = -\frac{\kappa^2}{T}\,\frac{d^2 T}{dt^2} = \lambda^4 \tag{4-19}$$

where the derivatives are now ordinary and the use of the separation constant, $\lambda^4$, which is determined from the boundary conditions, reflects the fact that the two sides of the equation are independent [107]. As

will be seen below, $\lambda$ (which has the reciprocal dimensions of x) becomes part of the arguments of the transcendental solutions to Equation (4-19). Equation (4-19) may be separated into the two equations

$$\frac{d^4Z}{dz^4} - \lambda^4 Z = 0 \tag{4-20}$$

and

$$\frac{d^2T}{dt^2} + \frac{\lambda^4}{\kappa^2} T = 0 \tag{4-21}$$

which are solved separately.

The solution to Equation (4-20) is given by

$$Z = A \sin \lambda z + B \cos \lambda z + C \sinh \lambda z + D \cosh \lambda z \tag{4-22}$$

The boundary conditions to be applied to Equation (4-22) are

$$Z(0) = Z(\ell) = \left.\frac{d^2Z}{dz^2}\right|_{z=0} = \left.\frac{d^2Z}{dz^2}\right|_{z=\ell} = 0 \tag{4-23}$$

since the bearings act as pin joints or hinges. Substitution of the first and third boundary conditions result in the two equations

$$B + D = 0 \tag{4-24}$$

$$B - D = 0 \tag{4-25}$$

which are satisfied only for B and D, both equal to zero. Applying the other two boundary conditions gives

$$A \sin \lambda\ell + C \sinh \lambda\ell = 0 \tag{4-26}$$

$$-A\lambda^2 \sin \lambda\ell + C\lambda^2 \sinh \lambda\ell = 0 \tag{4-27}$$

Since $\sinh \lambda\ell$ can never be equal to zero for positive (nonzero) values of $\lambda\ell$, Equations (4-26) and (4-27) are satisfied only for C equal to zero and $\sin \lambda\ell$ equal to zero (letting A be equal to zero results in a trivial solution). Thus, these equations are satisfied for an infinite set of values of $\lambda$ given by

$$\lambda_n = \frac{n\pi}{\ell}; \quad n = 1, 2, 3, \ldots \tag{4-28}$$

where the $\lambda_n$ are referred to as the characteristic values (or eigenvalues) of the rotor system. Substituting back into Equation (4-22) gives

$$Z_n = A_n \sin \frac{n\pi}{\ell} z = A_n \phi_n(z); \quad n = 1, 2, 3, \ldots \tag{4-29}$$

where the $\phi_n(z)$ are the characteristic functions (or eigenfunctions) of the rotor system. These eigenfunctions are orthogonal and are the same as the normalized mode shapes, $\phi_j(z)$, introduced in Equation (4-10), a few of which are represented in Figure 4-6. The total rotor response for any particular speed is generally written in rotating coordinates (leaving off the time varying functions) as a sum of the individual modal responses, $Z_n$, as

$$x(z) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{\ell} z = \sum_{n=1}^{\infty} A_n \phi_n(z) \tag{4-30}$$

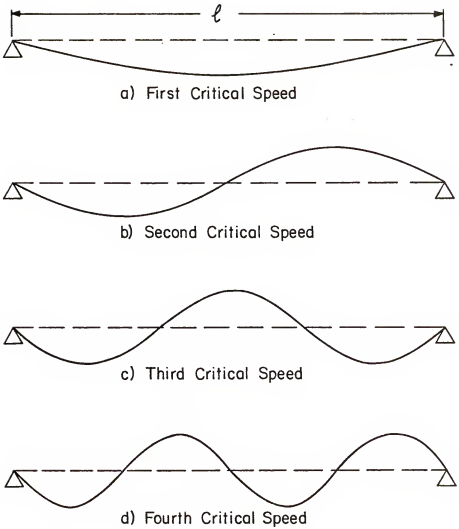



Figure 4-6 First four mode shapes for a uniform flexible rotor on rigid bearings

where the $A_n$ are coefficients which represent the extent to which each of the modes is excited, and are, in general, functions of the speed of rotation. Each $A_n$ reaches a maximum for a rotational speed equal to the critical speed for the $n^{th}$ mode, $\omega_n$.

The values of the critical speeds are determined by solving Equation (4-21), such that

$$T_n = F \sin \left[\frac{\lambda_n^2}{\kappa} t + \theta\right] = F \sin \left(\omega_n t + \theta\right); \quad n = 1, 2, 3, \ldots \tag{4-31}$$

where F and θ are (unimportant) constants which depend on the initial conditions and the $\omega_n$ are the rotor system resonances, which coincide with the critical speeds for an undamped system with negligible gyroscopic effects. Making the substitutions from Equations (4-17) and (4-28) the critical speeds are given by

$$\omega_n = \frac{n^2\pi^2}{\ell^2}\sqrt{\frac{EI}{\rho A}}; \quad n = 1, 2, 3, \ldots \tag{4-32}$$

For a uniform rotor with different boundary conditions, the solution is similarly straightforward. However, for nonuniform rotors, the solutions are much more involved and numerical methods are generally used.

As discussed above, the orthogonality of these characteristic functions permits them to be used to represent any well-behaved function, using the form of Equation (4-11). Particular functions of

interest include rotor mass unbalance and rotor deflection (or vibration amplitude). The rotor mass unbalance may be represented as an eccentricity of the mass centerline, with respect to the bearing centerline. This eccentricity is given as a complex function of z by

$$\hat{a}(z) = a_u(z) + ia_v(z) \tag{4-33}$$

where $a_u(z)$ and $a_v(z)$ are the components of the eccentricity in rotating coordinates, and where u and v are the system of rotating coordinates as illustrated in Figure 2-18. The complex function $\hat{a}(z)$ can be expanded in a modal (characteristic function) series as

$$\hat{a}(z) = \sum_{j=1}^{\infty} \phi_j(z)\hat{a}_j = \sum_{j=1}^{\infty} \phi_j(z)\left(a_{uj} + ia_{vj}\right) \tag{4-34}$$

where each of the complex coefficients, $\hat{a}_j$, can, in the form of Equation (4-12), be determined from

$$\hat{a}_j = \frac{1}{Q}\int_0^{\ell} A\rho(z)\hat{a}(z)\phi_j(z)dz \tag{4-35}$$

The superior caret is used to indicates a complex variable. Where all variables in an expression are complex and no ambiguity exists, the superior caret is dropped for convenience.

Similarly, the rotor whirl deformation can be represented in complex notation by

$$\hat{\eta}(z) = \eta_u(z) + i\eta_v(z) \tag{4-36}$$

which can also be expanded as a modal series

$$\hat{\eta}(z) = \sum_{j=1}^{\infty} \phi_j(z)\hat{\eta}_j = \sum_{j=1}^{\infty} \phi_j(z)\left(\eta_{uj} + i\eta_{vj}\right) \tag{4-37}$$

Modal balancing procedures are based on the orthogonality relations which require that the response for the $j^{th}$ mode, $\phi_j(z)\hat{\eta}_j$, is caused only by the $j^{th}$ component of mass eccentricity, $\phi_j(z)\hat{a}_j$. Thus, the individual modal responses are eliminated by removing the corresponding modal components of the mass eccentricity.

The modal response is related to the corresponding modal component of mass eccentricity in the form of the Jeffcott analysis as given by

$$\hat{\eta}_j = \frac{a_j\left(\omega/\omega_j\right)^2 e^{-i\beta_j}}{\sqrt{\left(1 - \omega^2/\omega_j^2\right)^2 - 4\left(\zeta_j\omega/\omega_j\right)^2}} \tag{4-38}$$

where $\omega_j$ is the undamped critical speed and $\zeta_j$ is the equivalent viscous damping ratio for the $j^{th}$ mode. Equation (4-28) may also be written in the form

$$\hat{\eta}_j = \frac{\hat{a}_j\omega^2}{\omega_j^2 - \omega^2 + 2i\zeta_j\omega_j\omega} \tag{4-39}$$

Note that the total synchronous response for any particular rotational speed is determined from the sum of the modal response for each mode as in Equation (4-37), where the $\hat{\eta}_j$ are complex, and are functions of

rotational speed only, and the $\phi_j$ are real and are functions of z only. In general, the $\omega_j$ and $\zeta_j$ are not known with sufficient accuracy to calculate $\hat{a}_j$ directly from $\hat{\eta}_j$ using Equation (4-38) or (4-39). However, from these equations it can be seen that $\hat{\eta}_j$ is directly proportional to $\hat{a}_j$ and that the proportionality constant, $\hat{\alpha}_j$, is a function of $\omega$ only. This premise, as applied to discrete unbalances, is central to influence coefficient balancing, as well as to the modal balancing procedures, as described below.

It has been shown in reference [35] that the addition of a discrete unbalance with corresponding masses $m_i$, located at axial positions $z_i$, has the effect of changing $\hat{a}_j$ by the amount $\hat{a}_j^*$, given by

$$\hat{a}_j^* = \sum_{i=1}^{n} \frac{m_i \hat{r}_i \phi_j(z_i)}{Q} \tag{4-40}$$

where the $\hat{r}_i$ are complex and represent the locations of the $m_i$ relative to the center of rotation, in rotating coordinates. The object of modal balancing is then to select the $m_i$, $\hat{r}_i$ and $z_i$ such that $\hat{a}_j^*$ exactly cancels $\hat{a}_j$, resulting in a rotor which is perfectly balanced for the $j^{th}$ mode. In practice, of course, $\hat{a}_j$ is never exactly cancelled and a rotor is never perfectly balanced. Rather, an acceptable residual $\hat{a}_j$ is generally specified, which is used as a balancing criterion. At the same time, it is necessary to avoid modifying the $\hat{a}_j$ for the previously balanced (usually lower) modes.

A variety of specific procedures have been used to perform modal balancing. These procedures generally fall into two groups referred to as N plane and (N+2) plane modal balancing, typified by

references [35] and [43], respectively. For both cases, it is necessary to determine the $\phi_j$ for all resonant modes below the maximum speed and one or two resonant modes above the maximum speed. Usually, this is done analytically, although it is suggested in reference [35] to do this experimentally. It is implied in this reference that by running the rotor at a speed very near to $\omega_j$ the rotor response will approximate $\phi_j$. However, the effect of other (usually higher) unbalanced modes may not be negligible. Thus, mode separation can be a problem. For N plane balancing the set of correction masses for the $N^{th}$ mode must satisfy

$$\sum_{i=1}^{n} m_i \hat{r}_i \phi_j\left(z_i\right) = 0 \; ; \qquad j = 1, 2, \ldots, N-1 \tag{4-41}$$

in order not to affect the corresponding residual $\hat{a}_j$ for the lower (previously balanced) modes. To eliminate $\hat{a}_N$ (i.e., to compensate for the effect of the $N^{th}$ mode), these correction masses must also satisfy

$$\sum_{i=1}^{n} m_i \hat{r}_i \phi_N\left(z_i\right) = - Q\hat{a}_N \tag{4-42}$$

where Q and $\hat{a}_N$ have been determined from trial mass runs. Generally, the $z_i$ are predetermined and the $m_i$ and $\hat{r}_i$ are combined into a single set of n complex unknowns (magnitude and angle), where all of the angles are assumed to be identical (planar mass distribution). Thus, Equations (4-41) ([N-1] real equations) and (4-42) (one complex equation) represent N equations with n unknowns ([n-1] real and one complex). To obtain a solution for these equations, n must be greater than or equal

to N. For N plane modal balancing, n is taken equal to N (i.e., to balance the $N^{th}$ mode requires N planes).

The (N+2) plane method is very similar except that the rotor is assumed to have been previously balanced as a rigid rotor and two additional constraint equations are added to preserve the rigid rotor balance

$$\sum_{i=1}^{n} m_i \hat{r}_i = 0 \tag{4-43}$$

and

$$\sum_{i=1}^{n} m_i \hat{r}_i z_i = 0 \tag{4-44}$$

At least two additional planes are required to satisfy Equations (4-43) and (4-44) along with Equations (4-41) and (4-42). Thus, (N+2) planes are needed to balance the $N^{th}$ mode for this case.

There has been considerable discussion as to whether the use of the two additional planes is justified for practical rotor balancing [35, 43]. Clearly, when balancing a rotor that has not been previously balanced as a rigid rotor, it is pointless to satisfy Equations (4-43) and (4-44). Thus, the question becomes whether it is justified to prebalance a flexible rotor as a rigid rotor. This will depend on two factors: (1) whether the rotor can be mounted on a balancing machine for rigid rotor balancing and then assembled on its own bearings in such a way as to retain the rigid rotor balance (otherwise there is no point in balancing as a rigid rotor); and (2) whether

the contributions of the unsatisfied equations (4-43) and (4-44) have a significant effect on the residual response of the rotor ("yes" according to reference [43] and "no" according to reference [35]). Certainly, examples can be (and have been) generated to support either position. Consequently, this discussion has not yet come to a satisfactory conclusion (i.e., satisfying all parties). The Unified Balancing Approach, as described in a later chapter, has been formulated as an N plane procedure. However, the additional constraints of Equations (4-43) and (4-44) can easily be included to result in an (N+2) plane procedure. Thus, the Unified Balancing Approach, while fundamentally different from modal balancing, can be configured to acquire the relative advantages of either N plane or (N+2) plane modal balancing.

The modal balancing theory, as described above, is based on two important assumptions: (1) the rotor system damping is negligible, and (2) the mode shapes are planar. These assumptions are not always satisfied in practice. As usually applied, the object of modal balancing is to eliminate bearing vibration. Consequently, the residual shaft vibration can be substantial (particularly if the bearings are located on, or near, shaft nodes). Thus, while the calculations required for modal balancing are relatively simple, the procedure requires considerable operator insight and is sensitive to the presence of measurement error, system damping and nonplanar mode shapes.

### 4.2.2 Influence Coefficient Balancing

As discussed above, the assumption that rotor response is proportional to unbalance is basic to virtually all balancing methods (for

both rigid and flexible rotors). Even for systems where substantial nonlinearity is observed, this assumption can be satisfied by balancing in a stepwise fashion with sufficiently small steps to approximate linear behavior. An additional assumption inherent in balancing methods, in general (with particular import to influence coefficient balancing), is that the effect of individual unbalances can be superposed to give the effect of a set of unbalances. This has been generally accepted as a fact for unbalances that are not excessively large. The premise behind influence coefficient balancing is that based on these two assumptions a rotor characterization, derived from a set of individual trial masses, can be used to define a combination of these masses which will eliminate (or minimize) the synchronous rotor response due to unbalance.

The influence coefficient balancing procedure, as currently employed, was originally developed by Goodman [51] and subsequently refined by Lund [55]. Using the complex notation introduced in Chapter 2, the synchronous rotor vibration, as measured in one stationary transverse direction, can be represented as

$$x = |x| \cos(\omega t + \beta) = x_c \cos(\omega t) - x_s \sin(\omega t) \qquad (4\text{-}45)$$

where

$$x_c = |x| \cos \beta \qquad (4\text{-}46)$$

$$x_s = |x| \sin \beta \qquad (4\text{-}47)$$

$|x|$ = vibration amplitude

$\beta$ = leading phase angle

$\omega$ = steady angular speed of shaft rotation

This vibration amplitude may also be represented with the conventional complex notation by

$$\hat{x} = x_c + ix_s \tag{4-48}$$

which is a short form representing

$$x = Re\left[\left(x_c + ix_s\right)e^{i\omega t}\right] \tag{4-49}$$

In a similar manner, rotor mass unbalance can be represented in complex form by

$$\hat{u} = u_c + iu_s \tag{4-50}$$

where $u_c$ and $u_s$ are the components of mass unbalance in rotating coordinates. The angle between the stationary coordinate axis along which x is measured and the rotating coordinate axis along which the $u_c$ component is taken is ($\omega t$). As virtually all variables in this section are complex, the superior carets are dropped for convenience.

For the balancing procedure, m balancing planes (axial locations) are used for the addition of correction masses. A total of n vibration measurements are taken to determine the response of the rotor. In general, these n measurements consist of data from $n_s$ sensors for each of the $n_\omega$ balancing speeds such that

$$n = n_s n_\omega \tag{4-51}$$

However, in theory, it is not necessary that data be used from the same number of sensors (or even the same sensors) for each balancing speed. For a linear system, the vibration measurements, represented by the column vector $\underset{\sim}{x}_w$ of length n, in response to masses placed in each of the m individual planes are given by

$$\underset{\sim}{x}_w = \underset{\sim}{A}\underset{\sim}{w} \tag{4-52}$$

where $\underset{\sim}{w}$ is a column vector of length m representing the masses in these planes and $\underset{\sim}{A}$ is an n by n matrix whose elements, $\alpha_{ij}$, are the influence coefficients relating the rotor response for the specified sensors and speeds to the balancing planes. Throughout this study the underscore tilde is used to indicate multidimensional vector (as distinguished from complex) and matrix quantities, with upper case letters used for matrices and lower case letters used for vectors. The $\alpha_{ij}$ are determined empirically through the application of known trial masses in each of the balancing planes (one plane at a time). In the simplest case, a single trial mass is used for each plane and

$$\alpha_{ij} = \frac{x_{ij} - x_{i0}}{T_j} \tag{4-53}$$

where $x_{i0}$ is the $i^{th}$ vibration reading with no trial masses installed, $x_{ij}$ is the $i^{th}$ vibration reading with a trial mass installed in the $j^{th}$ balancing plane, and $T_j$ is a complex value representing the amplitude

and angular location (in rotating coordinates) of this trial mass. Thus, (m+1) test runs (including the run with no trial masses) are required to provide sufficient data to calculate all $\alpha_{ij}$, where n measurements are taken for each test run.

Once the influence coefficients have been determined, the appropriate correction masses, $\underset{\sim}{w}_m$ can be calculated. The vibration which remains after the correction masses are installed (residual vibration) is given by

$$\underset{\sim}{x} = \underset{\sim}{x}_0 + \underset{\sim}{x}_m \tag{4-54}$$

where $\underset{\sim}{x}_0$ is the vector of complex vibration amplitudes measured prior to the installation of the correction masses, and $\underset{\sim}{x}_m$ is that due to the correction mass set which, from Equation (4-52), is related to the correction masses by

$$\underset{\sim}{x}_m = \underset{\sim}{A}\underset{\sim}{w}_m \tag{4-55}$$

Substituting Equation (4-55) into Equation (4-54) results in

$$\underset{\sim}{x} = \underset{\sim}{x}_0 + \underset{\sim}{A}\underset{\sim}{w}_m \tag{4-56}$$

The object of the balancing procedure is to minimize $\underset{\sim}{x}$.

For the case in which the number of measurements is equal to the number of balancing planes (n=m), Equation (4-56) may be solved directly with $\underset{\sim}{x}$ equal to zero to give

$$\underset{\sim}{w}_m = -\underset{\sim}{A}^{-1}\,\underset{\sim}{x}_0 \tag{4-57}$$

where $\underset{\sim}{A}^{-1}$ represents the matrix inverse of the square matrix $\underset{\sim}{A}$ (assuming $\underset{\sim}{A}$ is not singular).

However, for the case in which the number of measurements exceeds the number of balancing planes ($n > m$), $\underset{\sim}{x}$ cannot generally be reduced to zero. Instead, following the procedure of Lund [55], the object is to minimize the sum of squares, s, of the elements of $\underset{\sim}{x}$ where

$$s = \sum_{i=1}^{n} \left|x_i\right|^2 = \sum_{i=1}^{n} \bar{x}_i x_i = \bar{\underset{\sim}{x}}^T \underset{\sim}{x} \tag{4-58}$$

where the superscript bar represents the complex conjugate, the superscript T represents the transpose of a vector or matrix (together the bar and T represent the complex conjugate transpose of a vector or matrix), the $x_i$ represent the elements of $\underset{\sim}{x}$, and s is a positive real value.

In standard linear analysis fashion, s is minimized by the use of

$$\frac{\partial s}{\partial \bar{w}_{mj}} = 0 = \sum_{i=1}^{n} x_i \frac{\partial \bar{x}_i}{\partial \bar{w}_{mj}} = \sum_{i=1}^{n} x_i \bar{\alpha}_{ij} \quad (j=1, 2, \ldots, m) \tag{4-59}$$

where the $w_{mj}$ are the elements of $\underset{\sim}{w}_m$. Equation (4-56) may be rewritten for the individual complex elements of the vectors as

$$x_i = x_{0i} + \sum_{j=1}^{m} \alpha_{ij} w_{mj} \quad (i=1, 2, \ldots, n) \tag{4-60}$$

where the $x_{0i}$ are the elements of $\underset{\sim}{x}_0$. Substituting Equation (4-60) into Equation (4-59) gives

$$\sum_{i=1}^{n} \bar{\alpha}_{ij} \left( x_{0i} + \sum_{\ell=1}^{m} \alpha_{i\ell} w_{m\ell} \right) \quad (j=1, 2, \ldots, m) \tag{4-61}$$

Equation (4-61) may be rearranged and rewritten in matrix form yielding

$$\bar{\underset{\sim}{A}}^T \underset{\sim}{A} \underset{\sim}{w}_m + \bar{\underset{\sim}{A}}^T \underset{\sim}{x}_0 = 0 \tag{4-62}$$

Since $\bar{\underset{\sim}{A}}^T \underset{\sim}{A}$ is square, Equation (4-62) may be solved directly for $\underset{\sim}{w}_m$ (assuming $\bar{\underset{\sim}{A}}^T \underset{\sim}{A}$ is not singular), resulting in

$$\underset{\sim}{w}_m = -\left(\bar{\underset{\sim}{A}}^T \underset{\sim}{A}\right)^{-1} \bar{\underset{\sim}{A}}^T \underset{\sim}{x}_0 \tag{4-63}$$

This is the set of correction masses which minimizes the sum of squares (or, similarly, the root mean square) of the residual vibration. The expected residual vibration can be calculated by substituting Equation (4-63) (or the numerical result of that equation) into Equation (4-56).

While this least squares procedure provides for a minimization of the sum of the squares of the residual vibration amplitudes, it does not ensure that none of the individual residual vibration measurements remains large. In fact, it is not unusual for one or two of the vibration readings to remain quite large, while the remainder are reduced substantially. If this situation is undesirable and it is preferred to minimize the maximum residual amplitude (largest element of $\underset{\sim}{x}$), an iterative procedure may be used, which is due to Goodman [51]. In general, this procedure tends to equalize the elements of $\underset{\sim}{x}$ at the expense of the root mean square average of these elements.

To perform this procedure, Equation (4-58) is rewritten as

$$s_k = \sum_{i=1}^{n} \varepsilon_{ik} \left|x_i\right|_k^2 = \text{minimum} \qquad (k=0, 1, 2, \ldots) \qquad (4\text{-}64)$$

where k is the iteration number and $\varepsilon_{ik}$ is a weighting factor which is given by

$$\varepsilon_{ik} = \prod_{\ell=0}^{k-1} \frac{\left|x_i\right|_\ell}{(\text{rms})_\ell} = \frac{\left|x_i\right|_{k-1}}{(\text{rms})_{k-1}} \varepsilon_{i,k-1} \qquad (4\text{-}65)$$

where $\prod_{\ell=0}^{k-1}$ denotes a multiplication series, $(\text{rms})_\ell$ is the (real-valued) root mean square average of the elements of the vector $\underset{\sim}{x}$ from the $\ell^{th}$ iteration of the correction mass calculation, and $\varepsilon_{ik}$ is a positive real value. Thus

$$(\text{rms})_\ell = \sqrt{\frac{\sum_{i=1}^{n} \left|x_i\right|_\ell^2}{n}} = \sqrt{\frac{1}{n}\left(\bar{\underset{\sim}{x}}^T \underset{\sim}{x}\right)_\ell} \qquad (4\text{-}66)$$

The first iteration (k=0) is identical to the least squares calculation described above, Equation (4-63), which is obtained for

$$\varepsilon_{i0} = 1 \qquad (i=1, 2, \ldots, n) \qquad (4\text{-}67)$$

A result analogous to Equation (4-63) including the effect of the weighting factors is found in a similar manner as above. The final result is

$$\underset{\sim}{w}_{mk} = -\left(\bar{\underset{\sim}{A}}^T \underset{\sim}{E}_k \underset{\sim}{A}\right)^{-1} \bar{\underset{\sim}{A}}^T \underset{\sim}{E}_k \underset{\sim}{x}_0 \qquad (4\text{-}68)$$

where $\underset{\sim}{w}_{mk}$ is the $k^{th}$ iteration of $\underset{\sim}{w}_m$, and $\underset{\sim}{E}_k$ is an n-by-n diagonal matrix given by

$$\underset{\sim}{E}_k = \begin{vmatrix} \varepsilon_{1k} & & & 0 \\ & \varepsilon_{2k} & & \\ & & \ddots & \\ 0 & & & \varepsilon_{nk} \end{vmatrix} \tag{4-69}$$

A convergence limit can be specified for the iteration procedure such that when

$$\frac{\left| |x_i|_k - |x_i|_{k-1} \right|_{max}}{(rms)_k} \leqslant \text{convergence limit} \tag{4-70}$$

the iterations are considered to have converged.

It should be noted that the results of this least squares procedure (with or without the weighting factors) are dependent on the scaling of the vibration readings. Thus, to obtain meaningful results, it is generally advisable that the vibrations be measured in the same units and have the same order of magnitude. This can be accomplished by manipulation of calibration factors for the various sensors. By the same token, if it is desirable for the residual vibration at one or more particular sensors or balancing speeds to be less than that for the remaining sensors or speeds, a magnification factor can be applied to the corresponding elements of the weighting matrix before the application of Equation (4-68). In this way, a considerable amount

of flexibility can be introduced into the procedure for calculating correction masses [93]. Such a magnification factor is applied for a particular vibration reading by multiplying the element in the appropriate row of the matrix $\underset{\sim}{E}_k$ by this factor. This is equivalent to multiplying $\underset{\sim}{E}_k$ by the diagonal matrix, $\underset{\sim}{F}$, given by

$$\underset{\sim}{F} = \begin{vmatrix} f_1 & & & 0 \\ & f_2 & & \\ & & \ddots & \\ 0 & & & f_n \end{vmatrix} \tag{4-71}$$

where $f_i$ is the magnification factor for the $i^{th}$ row of $\underset{\sim}{E}_k$.

The influence coefficients may be most directly calculated using equation (4-53), once the appropriate vibration data has been collected. However, there are certain errors inherent in the measurement of real vibration data due to sensor and instrumentation inaccuracies and rotor system nonlinearities. Consequently, Lund [55] developed a procedure by which a second trial mass is used in the same plane and the two sets of trial mass data are used to optimize the influence coefficients and reduce the effect of these inherent measurement errors.

If the measurement errors for $x_{ij}$ and $x_{i0}$ are taken to be $\Delta x_{ij}$ and $\Delta x_{i0}$, respectively, Equation (4-53) is modified to give

$$\alpha_{ij} = \frac{\left(x_{ij} + \Delta x_{ij}\right) - \left(x_{i0} + \Delta x_{i0}\right)}{T_j} \tag{4-72}$$

where $\alpha_{ij}$ is now the "true" influence coefficient. A second trial mass, $T'_j$, which is different than $T_j$, is installed in the same balancing plane and the corresponding vibration, $x'_{ij}$, is measured. The corresponding influence coefficient, $\alpha'_{ij}$, is calculated from

$$\alpha'_{ij} = \frac{\left(x'_{ij} + \Delta x'_{ij}\right) - \left(x_{i0} + \Delta x_{i0}\right)}{T'_j} \tag{4-73}$$

where $\Delta x'_{ij}$ is the measurement error corresponding to $x'_{ij}$. Since $\alpha_{ij}$ and $\alpha'_{ij}$ are the same, the right sides of Equations (4-72) and (4-73) can be equated and rearranged to yield

$$\Delta x_{i0} = b_{ij} + c_j \Delta x_{ij} + d_j \Delta x'_{ij} \tag{4-74}$$

where

$$b_{ij} = c_j\left(x_{ij} - x_{i0}\right) + d_j\left(x'_{ij} - x_{i0}\right) \tag{4-75}$$

$$c_j = \frac{T'_j}{T_j - T'_j} \tag{4-76}$$

$$d_j = \frac{T_j}{T_j - T'_j} \tag{4-77}$$

and all values are complex.

The errors may then be determined by minimizing $s_{ij}$ where

$$s_{ij} = e_{ij}\left|\Delta x_{i0}\right|^2 + \left|\Delta x'_{ij}\right|^2 + \left|\Delta x_{ij}\right|^2 = e_{ij}\Delta x_{i0}\overline{\Delta x_{i0}} + \Delta x_{ij}\overline{\Delta x_{ij}} + x'_{ij}\overline{\Delta x'_{ij}} \tag{4-78}$$

and $s_{ij}$ and $e_{ij}$ are real values. Since $\Delta x_{i0}$ is used for $s_{ij}$ for all m values of j, the weighting factor $e_{ij}$ is used to fractionalize the $\Delta x_{i0}$ term among the corresponding m equations. Consequently, for any single balancing sensor, the $e_{ij}$ must sum to one. That is,

$$\sum_{j=1}^{m} e_{ij} = 1 \qquad (i=1, 2, \ldots, n) \tag{4-79}$$

Then, the $s_{ij}$ are minimized by taking

$$\frac{\partial s_{ij}}{\partial \left(\overline{\Delta x_{ij}}\right)} = 0 = e_{ij}\Delta x_{i0} \frac{\partial \left(\overline{\Delta x_{i0}}\right)}{\partial \left(\overline{\Delta x_{ij}}\right)} + \Delta x_{ij} \tag{4-80}$$

$$\frac{\partial s_{ij}}{\partial \left(\overline{\Delta x'_{ij}}\right)} = 0 = e_{ij}\Delta x_{i0} \frac{\partial \left(\overline{\Delta x_{i0}}\right)}{\partial \left(\overline{\Delta x'_{ij}}\right)} + \Delta x'_{ij} \tag{4-81}$$

The partial derivatives on the right sides of Equations (4-80) and (4-81) may be evaluated from the complex conjugate of Equation (4-74) to yield

$$\frac{\partial \left(\overline{\Delta x_{i0}}\right)}{\partial \left(\overline{\Delta x_{ij}}\right)} = \overline{c_j} \tag{4-82}$$

$$\frac{\partial \left(\overline{\Delta x_{i0}}\right)}{\partial \left(\overline{\Delta x_{ij}}\right)} = \overline{d_j} \tag{4-83}$$

Substituting Equations (4-82) and (4-83) into Equations (4-80) and (4-81), respectively, gives

$$e_{ij}\overline{c_j}\Delta x_{i0} + \Delta x_{ij} = 0 \tag{4-84}$$

$$e_{ij}\overline{d_j}\Delta x_{i0} + \Delta x'_{ij} = 0 \tag{4-85}$$

Equations (4-84) and (4-85) may be combined to give

$$\overline{d_j}\Delta x_{ij} = \overline{c_j}\Delta x'_{ij} \tag{4-86}$$

Then, Equation (4-86) may be solved for either $\Delta x_{ij}$ or $\Delta x'_{ij}$ and substituted into Equation (4-74) to yield the alternative forms

$$\Delta x_{i0} = b_{ij} + \frac{|c_j|^2 + |d_j|^2}{\overline{c_j}} \Delta x_{ij} \tag{4-87}$$

$$\Delta x_{i0} = b_{ij} + \frac{|c_j|^2 + |d_j|^2}{\overline{d_j}} x'_{ij} \tag{4-88}$$

Solving Equations (4-87) and (4-88) for $\Delta x_{ij}$ and $\Delta x'_{ij}$, respectively, yields

$$\Delta x_{ij} = \frac{\overline{c_j}}{|c_j|^2 + |d_j|^2} \left(\Delta x_{i0} - b_{ij}\right) \tag{4-89}$$

$$\Delta x'_{ij} = \frac{\overline{d_j}}{|c_j|^2 + |d_j|^2} \left(\Delta x_{i0} - b_{ij}\right) \tag{4-90}$$

Substituting either Equation (4-89) into Equation (4-84), or Equation (4-90) into Equation (4-85), and solving for $e_{ij}$ yields

$$e_{ij} = \frac{b_{ij} - \Delta x_{i0}}{\left(\left|c_j\right|^2 + \left|d_j\right|^2\right)\Delta x_{i0}} \tag{4-91}$$

By applying the constraint of Equation (4-79) to Equation (4-91), $e_{ij}$ can be eliminated. Then solving for $\Delta x_{i0}$ gives

$$\Delta x_{i0} = \frac{\sum_{j=1}^{m} \frac{b_{ij}}{\left|c_j\right|^2 + \left|d_j\right|^2}}{1 + \sum_{j=1}^{m} \frac{1}{\left|c_j\right|^2 + \left|d_j\right|^2}} \tag{4-92}$$

Substituting Equations (4-75) through (4-77) into Equation (4-92) results in an equation for $\Delta x_{i0}$ as a function of known quantities only.

$$\Delta x_{i0} = \frac{\sum_{j=1}^{m} \frac{\overline{T_j} - \overline{T_j'}}{\left|T_j\right|^2 + \left|T_j'\right|^2} \left[T_j\left(x_{ij}' - x_{i0}\right) - T_j'\left(x_{ij} - x_{i0}\right)\right]}{1 + \sum_{j=1}^{m} \frac{\left|T_j - T_j'\right|^2}{\left|T_j\right|^2 + \left|T_j'\right|^2}} \tag{4-93}$$

Similarly, equations for $\Delta x_{ij}$ and $\Delta x_{ij}'$ can be formed from Equations (4-87) and (4-88) to give

$$\Delta x_{ij} = \frac{\overline{T_j'}}{\left|T_j\right|^2 + \left|T_j'\right|^2}\left[T_j\left(x_{ij}' - x_{i0} - \Delta x_{i0}\right) - T_j'\left(x_{ij} - x_{i0} - \Delta x_{i0}\right)\right] \tag{4-94}$$

$$\Delta x'_{ij} = \frac{-\overline{T}_j}{|T_j|^2 + |T'_j|^2}\left[T_j\left(x'_{ij}-x_{i0}-\Delta x_{i0}\right) - T'_j\left(x_{ij}-x_{i0}-\Delta x_{i0}\right)\right] \tag{4-95}$$

Then, by substituting either Equation (4-94) into Equation (4-72), or Equation (4-95) into Equation (4-73), the relation for the adjusted value of the influence coefficient is obtained in the form

$$\alpha_{ij} = \frac{1}{|T_j|^2 + |T'_j|^2}\left[\overline{T}_j\left(x_{ij}-x_{i0}-\Delta x_{i0}\right) + \overline{T}'_j\left(x'_{ij}-x_{i0}-\Delta x_{i0}\right)\right] \tag{4-96}$$

Also, the corresponding element of the uncorrected rotor data vector, $x_{i0}$, is modified by adding $\Delta x_{i0}$ before performing any correction mass calculations.

Often in practice, the second trial mass, $T'_j$, is equal in magnitude and 180 degrees out of phase with the first trial mass, $T_j$, for the same balancing plane (i.e., $T'_j = -T_j$). In this case, Equation (4-96) reduces to

$$\alpha_{ij} = \frac{x_{ij} - x'_{ij}}{2T_j} \tag{4-97}$$

which is independent of $\Delta x_{i0}$. However, even for this special case, $x_{i0}$ is modified by adding the correction $\Delta x_{i0}$, and thus the calculated correction masses are affected by $\Delta x_{i0}$.

This procedure permits both the optimization of influence coefficients and the evaluation of measurement errors. This is useful

not only for improving the balancing calculation, but also for identifying erroneous data. If two trial mass runs per balancing plane are used, but, for some reason, it is not desired to optimize the influence coefficients or evaluate the measurement errors, the $\Delta x_{i0}$ are assumed equal to zero and Equation (4-96) is reduced to give the equation

$$\alpha_{ij} = \frac{\overline{T_j}\left(x_{ij} - x_{i0}\right) + \overline{T_j'}\left(x_{ij}' - x_{i0}\right)}{\left|T_j\right|^2 + \left|T_j'\right|^2} \tag{4-98}$$

for the calculation of influence coefficients. In addition, the $x_{i0}$ are not modified prior to the correction mass calculations.

A procedure has been developed by Larsson [102] for calculating influence coefficients from any number of balancing runs in which one or more trial (or correction) masses has been added to the rotor using a statistical regression algorithm. This procedure allows the values of the influence coefficients to be continually refined. However, the uncorrected rotor data are not modified to account for measurement error as with Lund's method [55].

As discussed above, the influence coefficient procedure is based on the assumptions of rotor linearity and superposition of rotor unbalance. An additional assumption inherent in the matrix manipulations described above is that the balancing planes are independent. That is, that the influence coefficient matrix, $\underset{\sim}{A}$, is of rank m (i.e., $\underset{\sim}{A}$ is well conditioned). This assumption is often, unknowingly, violated. The result can be the calculation of large correction masses in the nonindependent planes, which essentially cancel each other and

cannot practically be installed. This potential problem is discussed in more detail below and a proposed solution is described in Chapter 5.

When applying the influence coefficient procedure, some prior knowledge of the dynamics of the rotor system is useful when choosing the location and number of balancing planes and sensors, choosing the balancing speeds, and estimating safe vibration limits. The balancing procedure is begun by taking uncorrected rotor data. Then, either one or two trial masses are installed in each balancing plane (one at a time) and trial mass data is taken. In general, the trial masses are removed. However, if the software is designed to accommodate it, the trial masses may be left in [93]. The influence coefficient matrix, one or more expected sets of correction masses and the corresponding expected residual vibration are then calculated, as outlined above. Except for fairly simple cases, a computer is required to perform these calculations. A set of correction masses is chosen and installed and check balance data are taken. In many cases, this is the end of the balancing procedure. However, if the check balance data is not satisfactory, or if vibration must be controlled at a higher speed, it is necessary to perform another balancing run and the procedure described above is repeated.

Under certain circumstances, existing influence coefficients may be used for balancing a rotor, eliminating the requirement for trial mass runs [92]. In particular, trim balancing a rotor which is relatively linear at a speed (or speeds) at which it has been previously balanced can often be accomplished using existing influence

coefficients. Such trim balancing is sometimes necessary during the initial balancing procedure for a rotor (i.e., balancing in steps) and may also be used during maintenance of the same rotor system, if its dynamics have not changed substantially. Existing influence coefficients can also often be used for an identical rotor, if the dynamics are very similar (sometimes known as one-shot balancing [92]). This one-shot balancing procedure may be optimized by extracting influence coefficients from both trial and correction mass runs for many identical rotors and then using statistical methods for averaging these influence coefficients [102].

It is sometimes required to change balancing speeds between balancing runs, particularly when it is necessary to take data at a speed nearer to a critical speed in order to increase the sensitivity of the rotor to the residual unbalance. Since the influence coefficients change with speed, it is necessary to use a different set of influence coefficients, and, unless existing influence coefficients are available for the new balancing speed, a full set of trial mass runs are required. This can be very time-consuming, particularly if a large number of balancing planes are used. Influence coefficient balancing, like modal balancing, has a number of advantages and disadvantages, which are enumerated below.

### 4.2.3 Comparison of Modal and Influence Coefficient Balancing

Essentially, the influence coefficient method seeks those correction masses in a predetermined set of planes which will minimize

measured vibration (amplitudes) at a series of sensors and speeds. The balancing masses are predicted by influence coefficients, which relate vibration readings to mass additions. The influence coefficients are normally determined by a series of trial mass tests.

The modal method seeks to balance the rotor, one mode at a time, with a set of masses specifically selected not to disturb previously balanced lower modes. Sensitivity to this combination of masses as a set is determined empirically by a series of trial mass tests.

Specific advantages and disadvantages may be identified with each method, and according to requirements of a particular balancing process, these advantages and disadvantages take on different relative importance.

The principal advantages of modal balancing are

1. The number of sensitivity runs required at the highest balancing speed is minimized
2. Good sensitivity at the highest balancing speed can always be achieved
3. Balancing of a specific mode is permitted, while not affecting previously balanced (usually, lower) modes
4. It can be an entirely empirical procedure which requires only an understanding of the modal character of the response of a rotor system to unbalance.

Although, in present practice, it is most often used in conjunction with analytically determined mode shapes rather than in an empirical manner

The only trial mass runs that are required at the highest balancing speed involve the mass sets for the corresponding modal response. Individual trial mass runs are not required. Since the lower modes are not affected by a modal trial mass set for a higher mode, this mass set can be made large enough to assure good sensitivity at that higher mode. Eliminating the danger of upsetting the condition of previously balanced modes generally simplifies the balancing of higher modes. It is not necessary to have a prior knowledge of the dynamics of the rotor system apart from a basic understanding to assist in the satisfactory selection of locations for vibration sensors and balancing planes. Modal balancing does, of course, assume that the response of the rotor system is linear and that the vibration sensors are sufficiently accurate.

The principal disadvantages of modal balancing are

1. The assumption of planar modes of vibration inherent in modal balancing may not be valid for systems with substantial damping or bearing cross-coupling effects
2. For most (but not all) applications of modal balancing, accurate prior knowledge of the dynamics of the rotor system is required
3. Effective use requires a high degree of operator insight (i.e., requires a highly skilled operator)

4. Balancing results are generally based on the vibration measured by only one or two sensors for a particular mode, which may not result in a uniformly well-balanced rotor
5. Modal balancing has not usually been automated and does not easily lend itself to production applications
6. Often, trimming of lower modes while balancing higher modes is liable to adversely affect the higher modes (although, in theory, this can be avoided--with the addition of extra balancing planes)

Most applications of modal balancing use analytical procedures for selecting the sets of trial masses for correcting specific modes [42]. Not only does this complicate the balancing procedure, but any inaccuracies in this analytical model (which are inherent in any analytical representation of real components) tend to reduce the effectiveness of the balancing procedure. It is not necessary, however, to proceed in this way and several proponents of modal balancing employ empirical procedures to determine the modal trial mass sets [35]. However, this empirical procedure requires a substantial amount of operator insight into the modal character of the whirl of a rotor due to unbalance, which is not conducive to the development of efficient production balancing techniques for use by relatively unskilled personnel. Balancing of a rotor to eliminate vibration as measured by a single sensor may not result in a minimum vibration level for the rotor as a whole, particularly when dealing with distorted or non-planar mode

shapes. Difficulty may also occur as a result of the presence of vibration from other modes which have not been previously balanced, although methods have been developed for dealing with this problem [25]. Once again, however, operator insight is necessary.

The principal advantages of influence coefficient balancing are

1. It is an entirely empirical procedure which requires minimal prior knowledge of the dynamics of the rotor system
2. Full sensitivity information, if available, allows convenient balancing of any combination of critical speeds (modes), provided the rotor can be safely operated at the speeds for the measurement of the unbalanced whirl
3. It is readily computerized and automated
4. It provides for least-squares minimization of data from any number of vibration sensors
5. Data manipulation techniques have been developed which compensate for measurement errors
6. It does not require a high degree of operator insight

Other than a minimum knowledge of the rotor system, which is required for specifying the locations of the vibration sensors and balancing planes (for the application of correction masses), influence coefficient balancing is an entirely empirical procedure whose effectiveness is theoretically limited only by the range of linearity of the rotor system and the resolution of the vibration sensors. Influence

coefficient balancing provides for the simultaneous balancing of more than one mode, provided that the necessary sensitivity information (set of influence coefficients) is available. This can result in significant reduction in the required number of balancing runs, but does require measurement of the rotor vibration at speeds near the relevant critical speeds. It may not be possible to safely run through one of these critical speeds to make measurements at higher speeds.

As to the advantages of this method, the empirical nature of this procedure and the modest skills required of the operator, along with the form of the required calculations, make this procedure ideally suited for computerizing and automating. Least-squares minimization of vibration data is used to permit the application of balancing criteria based on reducing the vibration at a number of points in the rotor system for any number of modes. This feature of influence coefficient balancing can often be used to compensate for distorted or non-planar mode shapes and the effect of other unbalanced modes. Additional data manipulation techniques are available, which are designed to compensate for measurement errors inherent in any realistic application of rotor balancing.

Turning to the principal disadvantages of influence coefficient balancing

1. A significant number of runs is required to obtain rotor sensitivity data at the highest balancing speed
2. When the lowest modes are the most lightly damped, it is often difficult to get accurate sensitivity data near the highest modes

3. Data must be taken at previously balanced modes in order not to adversely affect the condition of these modes when balancing other modes
4. The least-squares procedure often results in the deterioration of previously balanced modes, unless these modes are heavily weighted, in which case the other modes may receive insufficient emphasis
5. The inadvertent use of nonindependent balance planes often results in the calculation of impractical (and generally inappropriate) results, such as large mutually counteracting correction masses

Since individual trial mass runs must be made at the highest balancing speed (as at all balancing speeds) when using influence coefficient balancing, the total number of runs required at this highest speed are substantially more than for modal balancing. In addition, the sizes of the individual trial masses to be used at the higher balancing speeds may be limited by the response of lightly damped lower modes to these trial masses. The initial unbalance present for the higher modes may make it impossible to acquire sensitivity data, using a restricted trial mass, at a speed close enough to a higher critical speed to produce a significant effect for this trial mass. That is, if the largest possible trial mass is small compared to the initial unbalance at the higher mode, the resolution of the sensitivity data acquired for this trial mass may be substantially reduced. When balancing a particular mode, after having previously balanced a number of other modes, a large quantity of data

may be required which can ultimately affect the efficiency of the balancing procedure in terms of both convenience and accuracy.

The inadvertent use of non-independent balance planes can result in singular, or near singular, influence coefficient matrices. In the case of a singular matrix, a computer program makes this condition known to the operator who can then change the balancing configuration. Due to noise in the vibration data signals and the effect of other unbalanced modes, it is virtually impossible for the influence coefficient matrix to be exactly singular. However, this matrix can be near singular (i.e., having a very small determinant), in which case the data would not be rejected by a computer program, and would often result in the calculation of fairly large correction masses which essentially cancel each other, insofar as their effect on the mode being balanced is concerned. A balance plane optimization procedure is described in the next chapter which can be used to detect the existence of an ill-conditioned (nearly singular) influence coefficient matrix and eliminate the redundant balance planes.

Neither modal nor influence coefficient balancing has gained general acceptance due to their inherent disadvantages. For this reason, the Unified Balancing Approach has been conceived and developed, combining the advantages and eliminating the disadvantages of modal and influence coefficient balancing. The Unified Balancing Approach is described in detail in a later chapter.

# CHAPTER 5
# ELIMINATION OF NON-INDEPENDENT BALANCE PLANES IN INFLUENCE COEFFICIENT BALANCING

As discussed in the previous chapter, the influence coefficient balancing calculations are prone to certain numerical difficulties when non-independent balance planes are inadvertently used, resulting in impractical, artificially large, mutually counteracting correction masses being calculated. This occurs as a result of the influence coefficient matrix being ill conditioned. That is, at least one of the columns of the matrix is close to being linearly dependent on the remaining columns.

As described in the previous chapter, this influence coefficient matrix represents a measure of the sensitivity of the rotor response for a unit trial mass placed at an arbitrary angular position in each of a number of axial locations. Each of these locations or balance planes is represented by a column of complex numbers (representing magnitude and angular orientation) in this matrix. For each trial mass, a number of vibration measurements are taken. This influence coefficient matrix may be either square or rectangular (with the number of rows exceeding the number of columns). It is proposed in this chapter that it is possible to anticipate and avoid this problem by evaluating the independence of the columns of the influence coefficient matrix. If this matrix is found to be ill conditioned, it is necessary to identify and eliminate the "least

independent" of the balancing planes. It is proposed that this be done by means of a Gram-Schmidt orthogonalization procedure for optimizing balance plane selection. The theoretical development of this original procedure is presented in this chapter along with the step-by-step algorithm required for its implementation.

## 5.1 Analytical Development

The influence coefficient balancing procedure involves the solution of the complex valued matrix equation

$$\underset{\sim}{A}\underset{\sim}{x} = -\underset{\sim}{b} \tag{5-1}$$

where $\underset{\sim}{A}$ is the influence coefficient matrix, $\underset{\sim}{b}$ is a column vector containing the unbalanced rotor vibration data and $\underset{\sim}{x}$ is the unknown column vector of the correction masses. As in the previous chapter, the underscore tilde is used to indicate multidimensional vector or matrix quantities. Virtually all variables in this chapter are complex. Thus, the superior caret is not used.

The influence coefficient matrix may be either square or rectangular. If this matrix is square, the solution of Equation (5-1), whether by inversion or decomposition, is straightforward. In the case of a rectangular matrix, a least-squares procedure is used to calculate the correction masses, as described in the previous chapter.

When correction planes are chosen that are not all clearly independent, certain problems can occur with the solution. To demonstrate what can occur, the following presentations of the matrix $\underset{\sim}{A}$ and the vector $\underset{\sim}{b}$ are proposed:

$$\underset{\sim}{A} = \begin{bmatrix} a & ca \\ b & cb + \varepsilon_1 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{b} = \begin{bmatrix} da \\ db + \varepsilon_2 \end{bmatrix} \tag{5-2}$$

for a simple case for which $\underset{\sim}{A}$ is square and of rank 2 (if $\varepsilon_1 \neq 0$). The value of $\varepsilon_1$ determines the extent of ill conditioning of the matrix $\underset{\sim}{A}$. Substituting Equations (5-2) into Equation (5-1) and solving for x yields

$$\underset{\sim}{x} = \begin{bmatrix} c\dfrac{\varepsilon_2}{\varepsilon_1} - d \\ \\ -\dfrac{\varepsilon_2}{\varepsilon_1} \end{bmatrix} \tag{5-3}$$

The term (-d) in Equation (5.3) represents the proper correction masses based on the first column of $\underset{\sim}{A}$ and the first element of $\underset{\sim}{b}$. The remaining terms in Equations (5-3) represent the effect of the second column of $\underset{\sim}{A}$ and the second element of $\underset{\sim}{b}$ on the correction mass calculation. For small values of $\varepsilon_1$ (nearly singular matrix $\underset{\sim}{A}$) these terms may become unreasonably large. Clearly, the magnitude of these terms is also dependent on the value of $\varepsilon_2$. However, experimental results have indicated that the near singularity of the influence coefficient matrix is more often the determining factor, rather than the form of the vector $\underset{\sim}{b}$. Thus, if the nearly dependent columns of the influence coefficient matrix are eliminated, the resulting correction mass calculation is almost certain to give reasonable results. Consequently, the

first and most important step in the optimization procedure presented here is the elimination of the nearly dependent columns of the influence coefficient matrix. This will ensure a well-behaved solution to Equation (5-1) regardless of the form of the vector $\underset{\sim}{b}$.

A Gram-Schmidt orthogonalization procedure [108] is used to eliminate the nearly dependent columns from the influence coefficient matrix. In practice, the optimum balancing planes are those that are not only independent but result in the smallest possible correction masses. That is, if two balancing planes are specified and are found not to be independent, the better balancing plane to use would be the one which would require the smaller correction mass. In general, this would be indicated by the column of the matrix with the largest Euclidean norm. For this reason, the first step in this procedure is to express the influence coefficient matrix as a group of column vectors and to rearrange these vectors in order of decreasing Euclidean norm. These vectors are represented by $\underset{\sim}{u}_1, \underset{\sim}{u}_2, \ldots, \underset{\sim}{u}_m$ where $\underset{\sim}{u}_1$ has the largest Euclidean norm and m balancing planes are initially specified.

A basis (or partial basis) is then found for this group of vectors and those vectors that are found not to be "clearly" independent of the previously determined basis vectors are discarded. A "significance" criteria is used to evaluate independence of the vectors as described below. The first basis vector is found by normalizing $\underset{\sim}{u}_1$ by

$$\underset{\sim}{e}_1 = \frac{\underset{\sim}{u}_1}{||\underset{\sim}{u}_1||} \tag{5-4}$$

where $\underset{\sim}{e}_i$ represents the $i^{th}$ basis vector and $||\underset{\sim}{u}_i||$ represents the Euclidean norm of $\underset{\sim}{u}_i$. The orthogonal portion of $\underset{\sim}{u}_2$ is given by

$$\underset{\sim}{v}_2 = \underset{\sim}{u}_2 - \left(\bar{\underset{\sim}{e}}_1^T \underset{\sim}{u}_2\right)\underset{\sim}{e}_1 \tag{5-5}$$

where the superscript bar represents the complex conjugate, and the superscript T represents the transpose of a vector or matrix. The "significance factor" for $\underset{\sim}{u}_2$ is given by

$$S_{f2} = \frac{||\underset{\sim}{v}_2||}{||\underset{\sim}{u}_2||} \tag{5-6}$$

This "significance factor" (which must be, by definition, real, non-negative and no larger than one) is a measure of the independence of $\underset{\sim}{u}_2$ from $\underset{\sim}{e}_1$ (or $\underset{\sim}{u}_1$). A real "independence criterion," $\delta$, must be chosen as a minimum acceptable value for $S_{f2}$. From a limited number of test cases run thus far, an appropriate value of $\delta$ appears to be about 0.2. If $S_{f2}$ is less than $\delta$, then $\underset{\sim}{u}_2$ is eliminated and $\underset{\sim}{u}_3$ substituted for $\underset{\sim}{u}_2$ in Equations (5-5) and (5-6). However, if $S_{f2}$ is not less than $\delta$, the second basis vector is given by

$$\underset{\sim}{e}_2 = \frac{\underset{\sim}{v}_2}{||\underset{\sim}{v}_2||} \tag{5-7}$$

The orthogonal portion of $\underset{\sim}{u}_3$ is then computed from

$$\underset{\sim}{v}_3 = \underset{\sim}{u}_3 - \left(\bar{\underset{\sim}{e}}_1^T \underset{\sim}{u}_3\right)\underset{\sim}{e}_1 - \left(\bar{\underset{\sim}{e}}_2^T \underset{\sim}{u}_3\right)\underset{\sim}{e}_2 \tag{5-8}$$

and the corresponding "significance factor" from

$$S_{f3} = \frac{||v_3||}{||u_3||} \tag{5-9}$$

Then $S_{f3}$ is compared to $\delta$ and either $\underset{\sim}{u}_3$ is eliminated (if $S_{f3} < \delta$) or $\underset{\sim}{e}_3$ is computed from

$$\underset{\sim}{e}_3 = \frac{\underset{\sim}{v}_3}{||\underset{\sim}{v}_3||} \tag{5-10}$$

(if $S_{f3} \geqslant \delta$). This procedure is repeated for each $\underset{\sim}{u}_i$.

To summarize this procedure, after $\underset{\sim}{e}_1$ is calculated from Equation (5-4) (clearly, the vector with the largest Euclidean norm is always retained), the following sequence of calculations is performed for each $\underset{\sim}{u}_i$:

1. The orthogonal portion of $\underset{\sim}{u}_i$ is computed from

$$\underset{\sim}{v}_i = \underset{\sim}{u}_i - \sum_{j=1}^{\ell} \left(\underset{\sim}{e}_j^T \underset{\sim}{u}_i\right) \underset{\sim}{e}_j \tag{5-11}$$

   where $\ell$ basis vectors have thus far been determined ($\ell \leqslant i - 1$).

2. The "significance factor" for $\underset{\sim}{u}_i$ is computed from

$$S_{fi} = \frac{||\underset{\sim}{v}_i||}{||\underset{\sim}{u}_i||} \tag{5-12}$$

3. If $S_{fi}$ is not less than $\delta$, the next basis vector is computed from

$$\underset{\sim}{e}_{\ell+1} = \frac{\underset{\sim}{v}_i}{||\underset{\sim}{v}_i||} \tag{5-13}$$

If $S_{fi}$ is less than $\delta$, $\underset{\sim}{u}_i$ is eliminated. In either case, the above procedure is then repeated for the next $\underset{\sim}{u}_i$, until it has been completed for $\underset{\sim}{u}_m$.

The correction mass calculation is then performed using the remaining columns of the influence coefficient matrix. With the exception of the significance factors and Euclidean norms, all values used in this procedure are complex.

## 5.2 Implementation of the Balance Plane Optimization Procedures

A computer program has been created which takes, as input, an influence coefficient matrix ($\underset{\sim}{A}$) of any shape (up to forty rows by ten columns), up to five separate uncorrected rotor vectors ($\underset{\sim}{b}$) and a value for $\delta$. First, the correction masses are calculated for this full influence coefficient matrix (unless it has more columns than rows, in which case the excess number of columns are dropped from the end of the matrix) and each of the uncorrected rotor vectors. The procedure described above, in Section 5.1, is then used to eliminate the dependent, or nearly dependent, balancing planes. Using the remaining balancing planes, the correction masses are then recalculated for each of the uncorrected rotor vectors (again dropping any columns of the matrix exceeding the number of rows).

The actual correction mass calculations are performed using subroutines extracted from the IBM scientific subroutine package [109]

and modified for complex arithmetic. For a square matrix, x is calculated using Gaussian elimination with full pivoting. For a rectangular matrix, a least-squares calculation is used to calculate x, using Householder transformations [110]. All calculations are performed in single precision. Generally speaking, the experimental error encountered in the vibration data used for practical rotor balancing is such that any roundoff error would be relatively insignificant by comparison. The results of an initial evaluation of this procedure are presented in the next section of this chapter.

In practical applications, it might be more desirable to combine the non-independent columns of the influence coefficient matrix with the independent columns, rather than simply discarding the former. The result of this operation would be to distribute the correction among all of the originally specified balance planes while ensuring that the calculation of the correction masses is well behaved. It might also be desirable to evaluate the vector $\underset{\sim}{b}$ with regard to its effect on this calculation, as mentioned above. This would probably be useful only for cases with marginal "significance factors" (say between 0.1 and 0.3). This evaluation would involve representing $\underset{\sim}{b}$ in terms of the basis vectors, $\underset{\sim}{e}_j$, as

$$\underset{\sim}{b} = \sum_{j=1}^{\ell} \alpha_j \underset{\sim}{e}_j \tag{5-14}$$

where the $\alpha_j$ are the coefficients of the basis vectors. Since the basis vectors are orthonormal, these coefficients can be calculated

simply by taking the inner product of $\underset{\sim}{b}$ with the corresponding basis vector. In general, the more nearly equal in value that the $\alpha_j$'s are, the less well behaved is the correction mass calculation.

## 5.3 Validation of the Procedure and Computer Program

A number of test cases have been run to demonstrate the balance plane optimization procedure. The data used for these test cases is not actual test data, but rather has been generated artificially. For some of the test cases presented herein, the data have been intentionally arranged to provide a somewhat ill-conditioned influence coefficient matrix. However, this ill-conditioning is not unrepresentative of the level of ill-conditioning often encountered in real influence coefficient matrices.

Six test cases are presented. For each of these test cases, the computer program output is provided along with a brief discussion of the significance of the test case. The computer program output for the first test case is also annotated to serve as sample program output. The first two tests cases involve only real-valued data and a square influence coefficient matrix of order two. The data for the remaining four test cases are all complex-valued. The influence coefficient matrices for the third and fourth test cases are square and of order four, while those for the last two test cases are rectangular, with four rows and three columns.

The significance of each of the test cases is as follows:

Test Case No. 1: The two columns of the matrix are roughly in phase (the same sign) as are the calculated correction masses. It is not difficult to observe that the

two correction masses reinforce each other rather than cancel each other. This is reflected by the reasonable significance factor. For this case, the only effect of the balance plane optimization procedure is to reverse the order of the columns in the influence coefficient matrix (and consequently to reverse the order of the correction masses).

Test Case No. 2: As for the previous test case, the two columns of the matrix are roughly in phase. However, the initial two-plane solution for this test case consists of correction masses that are out of phase and somewhat larger than for the previous test case. This indicates that these masses are, at least to a certain extent, canceling each other. Thus, it is reasonable to suspect an ill-conditioned matrix. This is verified by the small significance factor in the program output. Clearly, the subsequent reduction in order of the influence coefficient matrix results in an improved solution with the total of the (absolute value of the) correction masses only about one-third of that for the original solution.

Test Case No. 3: This is a somewhat more realistic (though less easily comprehended) test case in that the data are complex valued and the influence coefficient matrix is of a higher order. For this test case, the matrix is fairly well conditioned, as indicated by the values of the significance factors. Although some

canceling of the correction masses appears to be in evidence, the amount is fairly small and the total masses are reasonable.

Test Case No. 4: For this case, the influence coefficient matrix is not as well conditioned as for the previous masses, particularly between planes (columns) two and three, and the total masses are somewhat larger than for the previous test case. In fact, the solutions for the modified case (with the original plane number two eliminated) reflect total masses less than half those for the original solution.

Test Case No. 5: The data for this test case are similar to those for Test Case No. 3 with the fourth column of the matrix removed to give a least-squares problem (rectangular matrix). As for Test Case No. 3, the solutions are reasonable, as reflected by the significance factors.

Test Case No. 6: The data for this test case were generated from the data for Test Case No. 4 by eliminating the fourth column of the influence coefficient matrix to result in a least-squares problem. As for Test Case No. 4, the solutions are generally large and tending to cancel. Based on the small significance factor for the second column, a reduced solution has been calculated by the program. Clearly, the reduced solution is a substantial improvement over the original solution with the total

masses being reduced by at least a factor of two, in all cases.

There are certainly still some questions to be answered with regard to the practical application of this balance plane optimization procedure. Specifically, these questions concern the effect on real rotor balancing test data, a refinement of the acceptable level of the significance factor and whether this acceptable level should be related to the order of the influence coefficient matrix. These questions can and will be answered as practical experience is gained with this procedure.

Table 5-1 Test case no. 1

INFLUENCE COEFFICIENT MATRIX (Original)

| Column I | | Column 2 | |
|---|---|---|---|
| Real | Imaginary | Real | Imaginary |
| 2.0000 | 0.0000 | I.0000 | 0.0000 |
| 4.0000 | 0.0000 | 5.0000 | 0.0000 |

FOR UNCORRECTED ROTOR DATA OF (Vector $\underset{\sim}{b}$)

| Real | Imaginary | Real | Imaginary |
|---|---|---|---|
| 2.0000 | 0.0000 | 6.0000 | 0.0000 |

THE CORRECTION WEIGHT SET IS (Vector $\underset{\sim}{x}$)

| Plane I | | Plane 2 | |
|---|---|---|---|
| -0.6667 | 0.0000 | -0.6667 | -0.0000 |

THE SIGNIFICANCE VALUE OF COLUMN I = 0.2363I170 ($S_{f2}$)

THE REMAINING 2 COLUMNS ARE 2 I

| Basis Vectors $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ | Real | Imaginary | Real | Imaginary |
|---|---|---|---|---|
| E I. = | 0.I96II6I | 0.0000000 | 0.9805807 | 0.0000000 |
| E 2. = | 0.98058I9 | 0.0000000 | -0.I96II56 | 0.0000000 |

INFLUENCE COEFFICIENT MATRIX
(Modified)

| | | | |
|---|---|---|---|
| I.0000 | 0.0000 | 2.0000 | 0.0000 |
| 5.0000 | 0.0000 | 4.0000 | 0.0000 |

FOR UNCORRECTED ROTOR DATA OF

| | | | |
|---|---|---|---|
| 2.0000 | 0.0000 | 6.0000 | 0.0000 |

THE CORRECTION WEIGHT SET IS
(Modified)

| | | | |
|---|---|---|---|
| -0.6667 | -0.0000 | -0.6667 | 0.0000 |

Table 5-2 Test case no. 2

```
                    INFLUENCE COEFFICIENT MATRIX

  2.0000            0.0000            2.0000            0.0000

  4.0000            0.0000            5.0000            0.0000

                    FOR UNCORRECTED ROTOR DATA OF
  2.0000            0.0000            6.0000            0.0000

                    THE CORRECTION WEIGHT SET IS

  1.0000            0.0000           -2.0000           -0.0000

  THE SIGNIFICANCE VALUE OF COLUMN 1=0.0830455

  THE REMAINING 1 COLUMNS ARE 2

  E 1 =
0.3713907           0.0000000         0.9284768         0.0000000

                    INFLUENCE COEFFICIENT MATRIX

  2.0000            0.0000

  5.0000            0.0000

                    FOR UNCORRECTED ROTOR DATA OF
  2.0000            0.0000            6.0000            0.0000

                    THE CORRECTION WEIGHT SET IS
 -1.1724            0.0000
```

Table 5-3 Test case no. 3

```
INFLUENCE COEFFICIENT MATRIX

1.0000    1.0000    2.0000    1.0000    3.0000    2.0000    2.0000    4.0000

1.0000    3.0000    4.0000    2.0000    2.0000    1.0000    3.0000    1.0000

2.0000    2.0000    2.0000    1.0000    4.0000    3.0000    2.0000    2.0000

3.0000    1.0000    3.0000    2.0000    4.0000    2.0000    3.0000    3.0000

                FOR UNCORRECTED ROTOR DATA OF
1.0000    2.0000    2.0000    3.0000    3.0000    4.0000    2.0000    4.0000

                THE CORRECTION WEIGHT SET IS
0.4200   -0.0600   -0.6300   -0.1100   -1.2700    0.0100    0.5500   -0.6000

                FOR UNCORRECTED ROTOR DATA OF
1.0000    3.0000    3.0000    1.0000    4.0000    1.0000    2.0000    5.0000

                THE CORRECTION WEIGHT SET IS
1.4100   -0.9200   -0.8100    0.0300   -1.0900    1.4700   -0.5500   -1.4000

                FOR UNCORRECTED ROTOR DATA OF
1.0000    1.0000    2.0000    2.0000    3.0000    3.0000    2.0000    2.0000

                THE CORRECTION WEIGHT SET IS
0.1800    0.2600   -0.3700    0.0100   -1.1300   -0.1100    0.6000   -0.4500

THE SIGNIFICANCE VALUE OF COLUMN  4 =  0.3741939

THE SIGNIFICANCE VALUE OF COLUMN  2 =  0.3832476

THE SIGNIFICANCE VALUE OF COLUMN  1 =  0.2923524

THE REMAINING  4 COLUMNS ARE   3  4  2  1
```

Table 5-3--Continued

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| E 1 = | | | | | | | | |
| 0.3779646 | 0.2519764 | 0.2519764 | 0.1259882 | 0.5039528 | 0.3779646 | 0.5039528 | 0.2519764 | |
| E 2 = | | | | | | | | |
| 0.6170054 | 0.5725167 | 0.5559109 | 0.1133692 | -0.2324078 | -0.5271680 | 0.0396792 | 0.1303756 | |
| E 3 = | | | | | | | | |
| 0.1514309 | -0.5831740 | 0.5509352 | 0.4091893 | -0.3737450 | 0.0579949 | 0.0612174 | 0.1385443 | |
| E 4 = | | | | | | | | |
| -0.2552076 | 0.1921562 | -0.1601291 | 0.3202546 | -0.3522791 | 0.0640522 | 0.6405130 | -0.4803851 | |

INFLUENCE COEFFICIENT MATRIX

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3.0000 | 2.0000 | 2.0000 | 4.0000 | 2.0000 | 1.0000 | 1.0000 | 1.0000 |
| 2.0000 | 1.0000 | 3.0000 | 1.0000 | 4.0000 | 2.0000 | 1.0000 | 3.0000 |
| 4.0000 | 3.0000 | 2.0000 | 2.0000 | 2.0000 | 1.0000 | 2.0000 | 2.0000 |
| 4.0000 | 2.0000 | 3.0000 | 3.0000 | 3.0000 | 2.0000 | 3.0000 | 1.0000 |

FOR UNCORRECTED ROTOR DATA OF

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.0000 | 2.0000 | 2.0000 | 3.0000 | 3.0000 | 4.0000 | 2.0000 | 4.0000 |

THE CORRECTION WEIGHT SET IS

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| -1.2700 | 0.0100 | 0.5500 | -0.6000 | -0.6300 | -0.1100 | 0.4200 | -0.0600 |

FOR UNCORRECTED ROTOR DATA OF

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.0000 | 3.0000 | 3.0000 | 1.0000 | 4.0000 | 1.0000 | 2.0000 | 5.0000 |

THE CORRECTION WEIGHT SET IS

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| -1.0900 | 1.4700 | -0.5500 | -1.4000 | -0.8100 | 0.0300 | 1.4400 | -0.9200 |

FOR UNCORRECTED ROTOR DATA OF

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.0000 | 1.0000 | 2.0000 | 2.0000 | 3.0000 | 3.0000 | 2.0000 | 2.0000 |

THE CORRECTION WEIGHT SET IS

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| -1.1300 | -0.1100 | 0.6000 | -0.4600 | -0.3700 | 0.0100 | 0.1800 | 0.2800 |

Table 5-4 Test case no. 4

```
INFLUENCE COEFFICIENT MATRIX

 1.0000    1.0000    3.0000    2.0000    3.0000    2.0000    2.0000    4.0000
 1.0000    3.0000    2.0000    1.0000    2.0000    1.0000    3.0000    1.0000
 2.0000    2.0000    4.0000    3.0000    4.0000    3.0000    2.0000    2.0000
 3.0000    1.0000    3.0000    2.0000    4.0000    2.0000    3.0000    3.0000

          FOR UNCORRECTED ROTOR DATA OF
 1.0000    2.0000    2.0000    3.0000    3.0000    4.0000    2.0000    4.0000

          THE CORRECTION WEIGHT SET IS
-0.4541    0.5245   -0.6973    1.9858   -0.2000   -2.4000    0.3784   -0.2703

          FOR UNCORRECTED ROTOR DATA OF
 1.0000    3.0000    3.0000    1.0000    4.0000    1.0000    2.0000    5.0000

          THE CORRECTION WEIGHT SET IS
 0.5108    0.0351   -0.3405    2.6432   -0.4000   -1.8000   -0.6757   -0.9459

          FOR UNCORRECTED ROTOR DATA OF
 1.0000    1.0000    2.0000    2.0000    3.0000    3.0000    2.0000    2.0000

          THE CORRECTION WEIGHT SET IS
-0.2486    0.6919   -0.1676    1.2054   -0.8000   -1.6000    0.5405   -0.2432

THE SIGNIFICANCE VALUE OF COLUMN  2 =  0.1104003

THE SIGNIFICANCE VALUE OF COLUMN  4 =  0.3741959

THE SIGNIFICANCE VALUE OF COLUMN  1 =  0.3749940

THE REMAINING  3 COLUMNS ARE   3  4  1
```

Table 5-4--Continued

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| E 1 = | | | | | | | | |
| 0.3779646 | 0.2519764 | 0.2519764 | 0.1259882 | 0.5039528 | 0.3779646 | 0.5039528 | 0.2519764 | |
| E 2 = | | | | | | | | |
| -0.0170054 | 0.5725167 | 0.5555109 | -0.1133692 | -0.2324078 | -0.5271680 | 0.0396792 | 0.1363756 | |
| E 3 = | | | | | | | | |
| 0.0926430 | -0.0887018 | 0.0019716 | 0.6603338 | -0.4848990 | -0.0591335 | 0.4789083 | -0.2818720 | |

INFLUENCE COEFFICIENT MATRIX

| | | | | | |
|---|---|---|---|---|---|
| 3.0000 | 2.0000 | 2.0000 | 4.0000 | 1.0000 | 1.0000 |
| 2.0000 | 1.0000 | 3.0000 | 1.0000 | 1.0000 | 3.0000 |
| 4.0000 | 3.0000 | 2.0000 | 2.0000 | 2.0000 | 2.0000 |
| 4.0000 | 2.0000 | 3.0000 | 3.0000 | 3.0000 | 1.0000 |

FOR UNCORRECTED ROTOR DATA OF

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.0000 | 2.0000 | 2.0000 | 3.0000 | 3.0000 | 4.0000 | 2.0000 | 4.0000 |

THE CORRECTION WEIGHT SET IS

| | | | | | |
|---|---|---|---|---|---|
| -0.8510 | -0.3901 | -0.0449 | -0.4063 | -0.3721 | 0.5040 |

FOR UNCORRECTED ROTOR DATA OF

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.0000 | 3.0000 | 3.0000 | 1.0000 | 4.0000 | 1.0000 | 2.0000 | 5.0000 |

THE CORRECTION WEIGHT SET IS

| | | | | | |
|---|---|---|---|---|---|
| -0.4428 | 0.8033 | -1.2363 | -1.0028 | 0.6282 | 0.0874 |

FOR UNCORRECTED ROTOR DATA OF

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.0000 | 1.0000 | 2.0000 | 2.0000 | 3.0000 | 3.0000 | 2.0000 | 2.0000 |

THE CORRECTION WEIGHT SET IS

| | | | | | |
|---|---|---|---|---|---|
| -0.8314 | -0.4115 | 0.2040 | -0.2718 | -0.1953 | 0.7163 |

Table 5-5 Test case no. 5

```
INFLUENCE COEFFICIENT MATRIX

1.0000    1.0000    2.0000    1.0000    3.0000    2.0000
1.0000    3.0000    4.0000    2.0000    2.0000    1.0000
2.0000    2.0000    2.0000    1.0000    4.0000    3.0000
3.0000    1.0000    3.0000    2.0000    4.0000    2.0000

                FOR UNCORRECTED ROTOR DATA OF
1.0000    2.0000    2.0000    3.0000    3.0000    4.0000    2.0000    4.0000
                THE CORRECTION WEIGHT SET IS
0.8791    0.0040   -0.7057   -0.7735   -0.8511    0.0224

                FOR UNCORRECTED ROTOR DATA OF
1.0000    3.0000    3.0000    1.0000    4.0000    1.0000    2.0000    5.0000
                THE CORRECTION WEIGHT SET IS
1.7113   -1.6154   -1.9731   -0.3824   -0.7296    0.7845

                FOR UNCORRECTED ROTOR DATA OF
1.0000    1.0000    2.0000    2.0000    3.0000    3.0000    2.0000    2.0000
                THE CORRECTION WEIGHT SET IS
0.5767    0.3944   -0.3259   -0.6038   -0.7527   -0.0277

THE SIGNIFICANCE VALUE OF COLUMN  2 =  0.5021183

THE SIGNIFICANCE VALUE OF COLUMN  1 =  0.3413752

THE REMAINING  3 COLUMNS ARE   3  2  1
```

Table 5-5--Continued

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| E 1 = | | | | | | | | |
| 0.3779646 | 0.2519764 | 0.2519764 | 0.1259882 | 0.5039528 | 0.3779646 | 0.5039528 | 0.2519764 | |
| E 2 = | | | | | | | | |
| -0.0530286 | -0.1156988 | 0.7761497 | 0.4001266 | -0.2747856 | -0.3278143 | 0.0337458 | 0.1928326 | |
| E 3 = | | | | | | | | |
| -0.4138237 | -0.1605951 | -0.1024485 | 0.3541311 | -0.2145739 | 0.2920165 | 0.5767730 | -0.4477047 | |

INFLUENCE COEFFICIENT MATRIX

| | | | | | |
|---|---|---|---|---|---|
| 3.0000 | 2.0000 | 2.0000 | 1.0000 | 1.0000 | 1.0000 |
| 2.0000 | 1.0000 | 4.0000 | 2.0000 | 1.0000 | 3.0000 |
| 4.0000 | 3.0000 | 2.0000 | 1.0000 | 2.0000 | 2.0000 |
| 4.0000 | 2.0000 | 3.0000 | 2.0000 | 3.0000 | 1.0000 |

FOR UNCORRECTED ROTOR DATA OF

| | | | | | | |
|---|---|---|---|---|---|---|
| 1.0000 | 2.0000 | 2.0000 | 3.0000 | 3.0000 | 4.0000 | 2.0000 | 4.0000 |

THE CORRECTION WEIGHT SET IS

| | | | | | |
|---|---|---|---|---|---|
| -0.8511 | 0.0224 | -0.7057 | -0.7735 | 0.8701 | 0.0040 |

FOR UNCORRECTED ROTOR DATA OF

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.0000 | 3.0000 | 3.0000 | 1.0000 | 4.0000 | 1.0000 | 2.0000 | 5.0000 |

THE CORRECTION WEIGHT SET IS

| | | | | | |
|---|---|---|---|---|---|
| -0.7296 | 0.7045 | -1.9731 | -0.3826 | 1.9113 | -1.6154 |

FOR UNCORRECTED ROTOR DATA OF

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1.0000 | 1.0000 | 2.0000 | 2.0000 | 3.0000 | 3.0000 | 2.0000 | 2.0000 |

THE CORRECTION WEIGHT SET IS

| | | | | | |
|---|---|---|---|---|---|
| -0.7527 | -0.0277 | -0.3259 | -0.6038 | 0.5767 | 0.3944 |

Table 5-6 Test case no. 6

```
INFLUENCE COEFFICIENT MATRIX

 1.0000    1.0000    3.0000    2.0000    3.0000    2.0000

 1.0000    3.0000    2.0000    1.0000    2.0000    1.0000

 2.0000    2.0000    4.0000    3.0000    4.0000    3.0000

 3.0000    1.0000    3.0000    2.0000    4.0000    2.0000

          FOR UNCORRECTED ROTOR DATA OF
 1.0000    2.0000    2.0000    3.0000    3.0000    4.0000    2.0000    4.0000

          THE CORRECTION WEIGHT SET IS
-0.6692    0.3538   -1.2846    1.7769    0.8385   -2.2077

          FOR UNCORRECTED ROTOR DATA OF
 1.0000    3.0000    3.0000    1.0000    4.0000    1.0000    2.0000    5.0000

          THE CORRECTION WEIGHT SET IS
 0.0846    0.5731   -0.9577    4.1115    0.0808   -4.3962

          FOR UNCORRECTED ROTOR DATA OF
 1.0000    1.0000    2.0000    2.0000    3.0000    3.0000    2.0000    2.0000

          THE CORRECTION WEIGHT SET IS
-0.4750    0.4250   -0.8500    0.8000    0.4500   -1.1000

THE SIGNIFICANCE VALUE OF COLUMN  2 =  0.1104003

THE SIGNIFICANCE VALUE OF COLUMN  1 =  0.4657584

THE REMAINING  2 COLUMNS ARE   3  1
```

Table 5-6--Continued

```
 E 1 =
0.3779646   0.2519764   0.2519764   0.1259882   0.5039528   0.3779646   0.5039528   0.2519764

 E 2 =
-0.2426524  -0.1928853  -0.0435543   0.8648742  -0.0497764  -0.0746651   0.3048841  -0.2302179

        INFLUENCE COEFFICIENT MATRIX

   3.0000      2.0000      1.0000      1.0000

   2.0000      1.0000      1.0000      3.0000

   4.0000      3.0000      2.0000      2.0000

   4.0000      2.0000      3.0000      1.0000

                 FOR UNCORRECTED ROTOR DATA OF
   1.0000      2.0000      2.0000      3.0000      3.0000      4.0000      2.0000      4.0000
                 THE CORRECTION WEIGHT SET IS
  -0.4580     -0.8542     -0.6028      0.7844

                 FOR UNCORRECTED ROTOR DATA OF
   1.0000      3.0000      3.0000      1.0000      4.0000      1.0000      2.0000      5.0000
                 THE CORRECTION WEIGHT SET IS
  -1.1286     -1.0159      0.5373      1.2740

                 FOR UNCORRECTED ROTOR DATA OF
   1.0000      1.0000      2.0000      2.0000      3.0000      3.0000      2.0000      2.0000
                 THE CORRECTION WEIGHT SET IS
   0.3637     -0.5226     -0.4835      0.8567
```

# CHAPTER 6
# THE UNIFIED BALANCING APPROACH

The Unified Balancing Approach (UBA) is an entirely new balancing procedure designed to incorporate the advantages of both the influence coefficient and modal balancing methods, while eliminating the disadvantages of both methods. That is, the Unified Balancing Approach uses a modal method of applying correction masses in modal sets using data derived in an empirical manner and requiring a minimum of prior knowledge of the dynamics of the rotor. Essentially, the technique involves the calculation of modal trial mass sets based on the influence coefficient procedure of taking trial mass data. Generally, these modal trial mass sets are determined such that they affect the mode of interest while not having any effect on the lower modes that have already been balanced. However, using the appropriate data, if available from previous tests, a modal trial mass set can be constructed that will have no effect on any general set of modes (hereafter referred to as the unaffected modes), which can include modes above and below the mode being excited. In general, the number of planes required for the modal trial mass set is one more than the number of modes which must not be affected. In addition, the Unified Balancing Approach is not restricted to planar modes.

In this chapter, the theoretical development of the UBA is presented along with the step-by-step procedure required for its

implementation. Also, an approximation method for reducing the number of required trial mass runs is introduced and outlined.

## 6.1 Analytical Development

The complex notation representing rotor vibration for the UBA is the same as that used previously for synchronous vibration. To review, the synchronous vibration, as measured in one stationary transverse direction, can be represented as

$$x = |x| \cos(\omega t + \beta) = x_c \cos(\omega t) - x_s \sin(\omega t) \tag{6-1}$$

where

$$x_c = |x| \cos \beta \tag{6-2}$$

$$x_s = |x| \sin \beta \tag{6-3}$$

$|x|$ = vibration amplitude

$\beta$ = leading phase angle

$\omega$ = steady angular speed of shaft rotation

This vibration may also be represented with the conventional complex notation by

$$\hat{x} = x_c + i x_s \tag{6-4}$$

which is a short form to represent the sensible and measurable quantity

$$x = Re\left\{\left(x_c + i x_s\right) e^{i\omega t}\right\}$$

as illustrated by Figure 2-18 Since virtually all variables in this chapter are complex, the superior carets are again dropped for convenience.

### 6.1.1 Calculation of Modal Trial Mass Sets

The first step in the Unified Balancing Approach involves the calculation of a set of complex relationships which define the construction of a modal trial mass set for the mode to be corrected which will not introduce fresh unbalance to the modes which have already been corrected [111]. Hereafter, these complex relationships are referred to as modal trial mass set ratios. A total of m planes in the rotor are used to make up this modal trial mass set so that the number of planes is one greater than the number of modes which are to be unaffected by this modal trial mass set.

A total of n vibration measurements are taken to determine the response of the rotor at the (m-1) critical speeds specified above, with at least one measurement being taken at each of these critical speeds. For a linear rotor system, these vibration measurements, represented by the column vector $\underset{\sim}{x}_w$ of length n, in response to masses placed in each of the m individual planes are given by

$$\underset{\sim}{x}_w = \underset{\sim}{A}\underset{\sim}{w} \tag{6-5}$$

where $\underset{\sim}{w}$ is a column vector of length m representing the masses in these planes and $\underset{\sim}{A}$ is an n-by-m matrix whose elements, $\alpha_{ij}$, are the influence coefficients relating the rotor response for these specified modes to the balancing planes. The $\alpha_{ij}$ are determined empirically through the

application of known trial masses in each of the balancing planes (one plane at a time). In the simplest case, a single trial mass is used for each plane and

$$\alpha_{ij} = \frac{x_{ij} - x_{i0}}{T_j} \tag{6-6}$$

where $x_{i0}$ is the $i^{th}$ vibration reading with no trial masses installed, $x_{ij}$ is the $i^{th}$ vibration reading with a trial mass installed in the $j^{th}$ balancing plane, and $T_j$ is a complex value representing the amplitude and angular location of this trial mass. Thus, (m+1) test runs (including the run with no trial masses) are required to provide sufficient data to calculate all $\alpha_{ij}$, where n measurements are taken for each test run. The influence coefficients also can be calculated using a measurement error compensation algorithm described in a previous chapter.

In order to fulfill the requirement that the modal trial mass set not affect the (m-1) specified modes, it is necessary that the relation

$$\underset{\sim}{x}_w = \underset{\sim\sim}{A}w = 0 \tag{6-7}$$

be satisfied. The value of n is restricted to

$$n \geqslant m - 1 \tag{6-8}$$

since at least one vibration reading is required for each of the (m-1) specified modes. To simplify the specification of the modal correction mass sets, a reference plane is chosen from the set of m planes in this

modal trial mass set. The choice of the reference plane is arbitrary and is used during the balancing procedure to represent the modal trial mass set. Equation (6-7) can then be rewritten as

$$\underset{\sim}{A}'\underset{\sim}{w}' + \underset{\sim}{\alpha}_R w_R = 0 \tag{6-9}$$

where $w_R$ represents the mass in the reference plane. In addition $\underset{\sim}{\alpha}_R$ is a column vector formed from that column of matrix $\underset{\sim}{A}$ which corresponds to the reference plane. Also, $\underset{\sim}{A}'$ is a n-by-(m-1) matrix which is formed by removing the column from $\underset{\sim}{A}$ that forms $\underset{\sim}{\alpha}_R$, and $\underset{\sim}{w}'$ is a column vector formed by removing the element $w_R$ from $\underset{\sim}{w}$. For example, if the reference plane is plane k, the $\underset{\sim}{\alpha}_R$ is a column vector identical with the $k^{th}$ column of $\underset{\sim}{A}$. Then $\underset{\sim}{A}'$ is the n-by-(m-1) matrix which is formed by removing the $k^{th}$ column from $\underset{\sim}{A}$, and $\underset{\sim}{w}'$ is a column vector, of order (m-1), formed by removing the element $w_R$ from the $k^{th}$ row of $\underset{\sim}{w}$.

For the reduced case when

$$n = m - 1 \tag{6-10}$$

the matrix $\underset{\sim}{A}'$ is a square n-by-n matrix and Equation (6-9) can be solved for $\underset{\sim}{w}'$ directly to give

$$\underset{\sim}{w}' = -w_R[\underset{\sim}{A}']^{-1}\underset{\sim}{\alpha}_R \tag{6-11}$$

where, of course, $w_R$ is a scalar. Equation (6-11) can be rewritten as

$$\frac{1}{w_R}\underset{\sim}{w}' = -[\underset{\sim}{A}']^{-1}\underset{\sim}{\alpha}_R = \underset{\sim}{r}_W \tag{6-12}$$

where $\underset{\sim}{r}_W$ is a column vector of order (m-1) containing the complex ratios that relate each of the masses in the modal trial mass set to the mass in the reference plane. In some cases, $\underset{\sim}{r}_W$ may have real elements indicating that all the corrections for the particular mode are located in a single diametral plane along the rotor. Then, when this modal trial mass set is applied, the size and angular location of the mass in the reference plane can be chosen arbitrarily (just as an individual trial mass is chosen during influence coefficient balancing) and the trial mass in each of the remaining planes is determined by multiplying the reference plane trial mass by the appropriate "modal trial mass ratio" (element of $\underset{\sim}{r}_W$). This trial mass set will then have no effect on the (m-1) modes specified above. The balancing calculations (outlined below) are performed using the reference plane trial mass to represent the modal trial mass set, and the calculated correction mass is, in actuality, the element of the correction mass set located in the reference plane. The remaining masses in the correction mass set are then calculated from the reference plane correction mass and $\underset{\sim}{r}_W$, as described above.

For the more general case when

$$n > m - 1 \tag{6-13}$$

a weighted least-squares minimization procedure, similar to those described by Goodman [51] and Lund [55] for correction mass calculation, is performed. In this case, it is not possible to choose a modal trial mass set such that $\underset{\sim}{x}_W$ in Equation (6-5) is identically equal to zero (all

elements of the vector equal to zero). Instead, the sum of the squares, $s_w$, of the elements of $\underset{\sim}{x}_w$ is minimized where

$$s_w = \sum_{i=1}^{n} |x_{wi}|^2 = \sum_{i=1}^{n} x_{wi}\bar{x}_{wi} \tag{6-14}$$

The $x_{wi}$ are the elements of $\underset{\sim}{x}_w$ and the superscript bar denotes the complex conjugate. Following the procedure of Lund [55], $s_w$ (which is real-valued) is minimized by

$$\frac{\partial s_w}{\partial \bar{w}_j} = 0 = \sum_{i=1}^{n} x_{wi}\frac{\partial \bar{x}_{wi}}{\partial \bar{w}_j} = \sum_{i=1}^{n} x_{wi}\bar{a}_{ij} \quad (j=1, \ldots, m) \tag{6-15}$$

where the reference plane is again excluded. Rewriting Equation (6-5) as

$$\underset{\sim}{x}_w = \underset{\sim}{A}'\underset{\sim}{w}' + w_R\underset{\sim}{\alpha}_R \tag{6-16}$$

and substituting the result into Equation (6-15) gives, in matrix form

$$\bar{\underset{\sim}{A}}'^T\left(\underset{\sim}{A}'\underset{\sim}{w}' + w_R\underset{\sim}{\alpha}_R\right) = 0 \tag{6-17}$$

where $\bar{\underset{\sim}{A}}'^T$ is the conjugate transpose of the matrix $\underset{\sim}{A}'$. Solving Equation (6-17) for $\underset{\sim}{w}'$ gives

$$\underset{\sim}{w}' = - w_R\left(\bar{\underset{\sim}{A}}'^T\underset{\sim}{A}'\right)^{-1}\bar{\underset{\sim}{A}}'^T\underset{\sim}{\alpha}_R \tag{6-18}$$

Dividing by $w_R$, which is a scalar, gives the solution for the ratio vector $\underset{\sim}{r}_W$ as

$$\underset{\sim}{r}_W = - \left(\underset{\sim}{\bar{A}}'^T \underset{\sim}{A}'\right)^{-1} \underset{\sim}{\bar{A}}'^T \underset{\sim}{\alpha}_R \tag{6-19}$$

This equation represents the choice of the modal trial mass set which has the minimum effect on the selected modes, in the least squares sense.

The anticipated response to this modal trial mass set for each of the vibration readings is calculated as

$$\underset{\sim}{x}'_W = \frac{1}{w_R} \underset{\sim}{x}_W = \underset{\sim}{A}' \underset{\sim}{r}_W + \underset{\sim}{\alpha}_R \tag{6-20}$$

where $x'_W$ is the vector of resultant vibration readings for a unit trial mass (at zero degrees) in the reference plane.

Although the modal trial mass set calculated from Equation (6-21) has the least overall effect on the n vibration readings, the effect on the individual vibration readings may be widely variable. If it is preferred to minimize the maximum effect on a vibration reading (largest element of $\underset{\sim}{x}_W$ or $\underset{\sim}{x}'_W$), an iteration procedure may be used, which is based on a procedure described by Goodman [51]. In general, this procedure tends to equalize the magnitude of the elements of $\underset{\sim}{x}'_W$ at the expense of the root mean square average of these elements.

To perform this proceudre, Equation (6-14) is rewritten as

$$s_{wk} = \sum_{i=1}^{n} \varepsilon_{ik} \left| x'_{wi} \right|_k^2 = \text{minimum } (k=0, 1, 2, \ldots) \tag{6-21}$$

where k is the iteration number and $\varepsilon_{ik}$ is a weighting factor which is given by

$$\varepsilon_{ik} = \prod_{\ell=0}^{k-1} \frac{|x'_{wi}|_\ell}{(rms)_\ell} = \frac{|x'_{wi}|_{k-1}}{(rms)_{k-1}} \varepsilon_{i,k-1} \tag{6-22}$$

where $\prod_{\ell=0}^{k-1}$ denotes a multiplication series, $(rms)_\ell$ is the (real-valued) root mean square of the elements of the vector $x'_w$ from the $\ell^{th}$ iteration of the modal trial mass set calculation, and $\varepsilon_{ik}$ is real. Thus

$$(rms)_\ell = \sqrt{\frac{\sum_{i=1}^{n} |x'_{wi}|_\ell^2}{n}} \tag{6-23}$$

The first iteration (k=0) is identical to the least-squares calculation described above, Equation [6-19), which is obtained for

$$\varepsilon_{i0} = 1 \qquad (i=1, 2, \ldots, n) \tag{6-24}$$

A result analogous to Equation (6-19) including the effect of the weighting factors is found in a similar manner as above. The final result is

$$\underset{\sim}{r}_{wk} = -\left(\underset{\sim}{\bar{A}}'^T \; \underset{\sim}{E}_k \; \underset{\sim}{A}'\right)^{-1} \; \underset{\sim}{\bar{A}}'^T \; \underset{\sim}{E}_k \; \underset{\sim}{\alpha}_R \tag{6-25}$$

where $\underset{\sim}{r}_{wk}$ is the $k^{th}$ iteration or $\underset{\sim}{r}_w$, and $\underset{\sim}{E}_k$ is a n-by-n diagonal matrix given by

$$\underset{\sim}{E}_k = \begin{bmatrix} \varepsilon_{1k} & & & 0 \\ & \varepsilon_{2k} & & \\ & & \ddots & \\ 0 & & & \varepsilon_{nk} \end{bmatrix} \tag{6-26}$$

A convergence limit can be specified for the iteration procedure such that when

$$\frac{\left|\,|X'_{wi}|_k - |X'_{wi}|_{k-1}\right|_{max}}{(rms)_k} \leqslant \text{convergence limit} \qquad (6\text{-}27)$$

the iterations are considered to have converged.

It should be noted that the results of this least-squares procedure (with or without the weighting factors) are dependent on the scaling of the vibration readings. Thus, to obtain meaningful results, it is generally advisable that the vibrations be measured in the same units and have the same order of magnitude. By the same token, if it is desirable for the effect of the modal trial mass set on a specific vibration reading, or readings, be less than the effect on the remaining vibration readings, a magnification factor can be applied to the corresponding elements of the weighting matrix before the application of Equation (6-25). In this way, a considerable amount of flexibility can be introduced into the procedure for calculating modal trial mass sets. Such a magnification factor is applied for a particular vibration reading by multiplying the appropriate row of the matrix $\underset{\sim}{E}_k$ by this factor.

### 6.1.2 Calculation of Modal Correction Mass Sets

After the modal trial mass set has been calculated for a particular mode to be balanced, vibration readings are taken at that mode for the uncorrected rotor system, and then with a modal trial mass set installed. A column vector, of length N, of modal influence

coefficients are calculated as described below, where N is the number of vibration readings taken at the mode of interest. The residual vibration of the rotor system, $\underset{\sim}{x}$, after the addition of a modal correction mass set, is given as

$$\underset{\sim}{x} = \underset{\sim}{x}_0 + w_m \underset{\sim}{\alpha}_m \tag{6-28}$$

where $\underset{\sim}{x}$ is a column vector of length N, with elements $x_{ij}$; $\underset{\sim}{x}_0$ is a similar column vector representing the vibration readings for the uncorrected rotor at the mode of interest; $\underset{\sim}{\alpha}_m$ is the modal influence coefficient vector; and $w_m$ (a scalar) is the element of the modal correction mass set located in the reference plane.

The aim of the rotor balancing procedure is to specify $w_m$ (and thus all elements of the modal correction mass set) to minimize $\underset{\sim}{x}$. For N equal to one, Equation (6-28) can be solved directly to give

$$w_m = - x_0 / \alpha_m \tag{6-29}$$

where $x_0$ and $\alpha_m$ are now scalars.

For the more general case, when N is not equal to one, a least-squares minimization procedure is performed, similar to that described above for the calculation of the modal trial mass sets. This procedure is somewhat simplified here, since only vectors and scalars are involved.

In this case, the sum of the squares of the residual amplitudes, s (which is real-valued), is minimized, where

$$s = \sum_{i=1}^{N} |x_i|^2 = \sum_{i=1}^{N} x_i \bar{x}_i \tag{6-30}$$

The partial derivative of s with respect to $\overline{w}_m$ is now set equal to zero to give

$$\frac{\partial s}{\partial \overline{w}_m} = \sum_{i=1}^{N} x_i \frac{\partial \overline{x}_i}{\partial w_m} = \sum_{i=1}^{N} x_i \overline{\alpha}_{mi} = 0 \tag{6-31}$$

where $\overline{\alpha}_{mi}$ is the complex conjugate of the $i^{th}$ element of the vector $\underset{\sim}{\alpha}_m$. Substituting Equation (6-28) into Equation (6-31) results in

$$\sum_{i=1}^{N} \overline{\alpha}_{mi}\left(x_{i0} + \alpha_{mi} \quad w_m\right) = \overline{\underset{\sim}{\alpha}}_m^T \quad \underset{\sim}{x}_0 + \overline{\underset{\sim}{\alpha}}_m^T \underset{\sim}{\alpha}_m w_m = 0 \tag{6-32}$$

where $x_{i0}$ is the $i^{th}$ element of the vector $\underset{\sim}{x}_0$. Solving Equation (6-32) for $w_m$ gives

$$w_m = -\left(\overline{\underset{\sim}{\alpha}}_m^T \underset{\sim}{x}_0\right) \Big/ \left(\overline{\underset{\sim}{\alpha}}_m^T \underset{\sim}{\alpha}_m\right) \tag{6-33}$$

where $\overline{\underset{\sim}{\alpha}}_m^T$ is the conjugate transpose of the vector $\underset{\sim}{\alpha}_m$ and the result is a division of scalar quantities.

In a similar fashion to that described above for the calculation of the modal trial mass sets, an iterative weighted least-squares minimization procedure can be used for calculating $w_m$. In this case Equation (6-30) is rewritten as

$$s_k = \sum_{i=1}^{N} \varepsilon_{ik} \left|x_i\right|_k^2 = \text{minimum } (k=0, 1, 2, \ldots) \tag{6-34}$$

where k is again the iteration number and $\varepsilon_{ik}$ is a real-valued weighting factor given by

$$\varepsilon_{ik} = \prod_{\ell=0}^{k-1} \frac{|x_i|_\ell}{(rms)_\ell} = \frac{|x_i|_{k-1}}{(rms)_{k-1}} \cdot \varepsilon_{i,k-1} \tag{6-35}$$

where $(rms)_k$ is real and is given by

$$(rms)_k = \sqrt{\frac{\sum_{i=1}^{N} |x_i|_k^2}{N}} \tag{6-36}$$

The first iteration (k=0) is identical to the least-squares calculation described above (Equation [6-33]) which is obtained for

$$\varepsilon_{i0} = 1; \qquad i=1, 2, \ldots, N \tag{6-37}$$

Minimizing the value of Equation (6-34) in the same manner as was done for Equation (6-30) yields

$$w_{mk} = \left(\bar{\alpha}_m^T E_k x_0\right) \Big/ \left(\bar{\alpha}_m^T E_k \alpha_m\right) \tag{6-38}$$

where $w_{mk}$ is the $k^{th}$ iteration for $w_m$ and $E_k$ is defined in Equation (6-26).

The discussion involving the scaling of vibration readings presented in the previous section applies to the use of Equations (6-33) and (6-38) as well.

### 6.1.3 Calculation of Influence Coefficients

The procedure for calculating and optimizing influence coefficients is essentially the same when using individual trial mass and modal trial mass sets. The only difference is that the complex

value of the reference plane trial mass (mass and angle) only is used to represent the entire modal trial mass set, while the individual trial mass values are each used in the appropriate calculations.

In general, placing a known trial mass, $T_j$, at a known angle in plane j (j=1, . . . , m for an individual trial mass; j=1 for modal trial mass sets) results in a new $i^{th}$ vibration reading, $x_{ij}$, giving the corresponding influence coefficient as

$$\alpha_{ij} = \frac{x_{ij} - x_{i0}}{T_j} \tag{6-39}$$

where $x_{i0}$ is the $i^{th}$ vibration reading for the undisturbed rotor. However, there are certain errors inherent in the measurement of real vibration data. Lund [55] defined a procedure by which a second trial mass is used in the same plane and the two sets of trial mass data are used to optimize the influence coefficients and reduce the effect of these inherent measurement errors. The equations derived by Lund to perform this optimization procedure are reproduced here and the derivations are presented in Chapter 4. The optimized influence coefficients are calculated as

$$\alpha_{ij} = \frac{1}{|T_j|^2 + |T_j'|^2} \left[ \overline{T}_j \left( x_{ij} - x_{i0} - \Delta x_{i0} \right) + \overline{T}_j' \left( x_{ij}' - x_{i0} - \Delta x_{i0} \right) \right] \tag{6-40}$$

where the primed quantities refer to the second trial mass run and $\Delta x_{i0}$, the measurement error for the undisturbed rotor vibration data, is given by

$$\Delta x_{i0} = \frac{\sum_{j=1}^{m} \frac{\overline{T}_j - \overline{T}'_j}{\left|T_j\right|^2 + \left|T'_j\right|^2} \left[T_j\left(x'_{ij} - x_{i0}\right) - T'_j\left(x_{ij} - x_{i0}\right)\right]}{1 + \sum_{j=1}^{m} \frac{\left|T_j - T'_j\right|^2}{\left|T_j\right|^2 + \left|T'_j\right|^2}} \tag{6-41}$$

When calculating modal correction masses (m=1), Equation (6-41) reduces to

$$\Delta x_{i0} = \frac{\left(T_1 - T'_1\right)\left[T_1\left(x_{i1} - x_{i0}\right) - T'_1\left(x_{i1} - x_{i0}\right)\right]}{\left|T_1\right|^2 + \left|T'_1\right|^2 + \left|T_1 - T'_1\right|^2} \tag{6-42}$$

The values for the $x_{i0}$ are subsequently modified to remove the measurement error by adding the $\Delta x_{i0}$ before calculating the modal correction mass set from Equation (6-29), Equation (6-33) or Equation (6-38). In addition, it should be noted that for the special case when

$$T'_j = -T_j \tag{6-43}$$

Equation (6-40) reduces to

$$a_{ij} = \frac{x_{ij} - x'_{ij}}{2T_j} \tag{6-44}$$

which is independent of $\Delta x_{i0}$. However, for this special case, $x_{i0}$ is still modified by adding the correction $\Delta x_{10}$.

## 6.2 Application of the Principle of Reciprocity for Reducing the Number of Trial Mass Runs

For some rotor systems, it is essential to minimize the number of test runs required for satisfactory balancing. For example, some large steam turbine/generator sets require several hours of operation to reach steady-state conditions after a system shutdown. Consequently, it is not unlikely that balancing of such a system using conventional procedures could take several days. The experimental results presented in Chapter 7 showed that the Unified Balancing Approach required fewer runs to achieve a given balance condition than modal or influence coefficient balancing. This difference is due entirely to the relative effectiveness of the balancing procedure rather than a reduction in the number of procedural steps.

When using the Unified Balancing Approach, the required minimum number of trial mass runs can be substantially reduced by approximating a number of the influence coefficients through a procedure, outlined in this section, which is based on Rayleigh's principle of reciprocity [112]. This principle, when applied to dynamic systems, states that the response at one point in a structure due to a harmonic excitation at a second point in the structure is identical (equal in amplitude and phase) to the response at the second point, if the same excitation is applied at the first point. Rayleigh showed that this principle is valid for damped as well as undamped systems (the damping need not be viscous). In terms of influence coefficients, this means that for two balancing planes, say A and B, with immediately adjacent displacement probes, the influence coefficient relating the sensor reading at A to a trial mass at B should be identical to the influence coefficient relating the sensor reading at

B to a trial mass at A. This, of course, requires that the sensor calibration factors be accurately determined.

For planar mode shapes, this reciprocity is clearly valid. However, if the mode shapes are not planar, due for example to the presence of bearing cross-coupling parameters, the principle of reciprocity might not apply.

It does not appear that the use of this principle of reciprocity would provide a substantial benefit to influence coefficient balancing. However, the use of this principle, together with an appropriate bookkeeping system, with the Unified Balancing Approach could be used to reduce the number of individual trial mass runs to a single run per mode. The procedure for applying the principle of reciprocity to the Unified Balancing Approach is as follows:

1. A displacement probe is located as near as possible to each balancing plane (the distance between the plane and probe is more critical for higher modes).
2. After any particular mode has been satisfactorily balanced (except for the highest mode of interest) by the use of a modal correction mass set, a single trial mass is installed in a balancing plane which is at or near an anti-node for the mode just balanced. Vibration data are taken for all displacement probes and the corresponding influence coefficients are calculated.
3. By reciprocity, these influence coefficients are the same as those which relate the vibration data at the

trial mass plane to unbalance masses located at the other balancing planes. It is only necessary to take into account differences in the calibration factors of the displacement probes and the radii at which the correction masses are applied.

4. The appropriate influence coefficients can then be chosen for calculating the modal trial mass sets, as required.

Normally, the balancing of a rotor through N critical speeds using the Unified Balancing Approach requires [N(N+1)/2]-1 individual trial mass runs. Using the reciprocity approach proposed herein, the number of individual trial mass runs is reduced to (N-2) (an individual trial mass run is not required for the first mode). Thus, a reduction of [N(N-1)/2]+1 trial mass runs is obtained. So, if four modes must be balanced, the required number of individual trial mass runs can be reduced by seven (from nine to two). For the same example, the total number of trial mass runs is reduced from thirteen to six which is a reduction of more than fifty percent. Thus, a very substantial reduction in required trial mass runs can be achieved using this reciprocity approach.

## 6.3 Procedure for Application of the Unified Balancing Approach

An on-line computer program and operating procedures for a laboratory prototype UBA balancing system have been implemented [113]. The program and system have been debugged, and balancing tests conducted. The results of these tests are discussed in a later chapter.

The operating procedure for this system, illustrated in the flowchart in Figure 6-1, is described here to give a clearer understanding of the process by which the Unified Balancing Approach is applied [114]. It should be kept in mind that the Unified Balancing Approach is inherently applications oriented. That is, this approach is designed to handle a number of non-ideal (but not uncommon) conditions, including the occurrence of non-planar or distorted mode shapes and the existence of measurement error and finite vibration data resolution.

The basic procedure for implementing the Unified Balancing Approach, where the paragraph numbers correspond to the step numbers on the flowchart in Figure 6-1, is as follows:

1. The balancing specifications for the rotor system are defined. These specifications include the number, types (e.g., displacement, velocity, acceleration or force) and calibration factors of the vibration sensors to be used and the specific sensors which are most sensitive for each of the modes to be balanced; the number of balancing planes and balancing mass hole specifications for each plane, as well as the specific planes which comprise each of the modal trial mass sets; the angular location of the trial masses and whether additional trial mass runs for reduction of measurement errors are to be used; the number of modes to be balanced and the speed at which vibration data are to be taken for each mode (these values can be

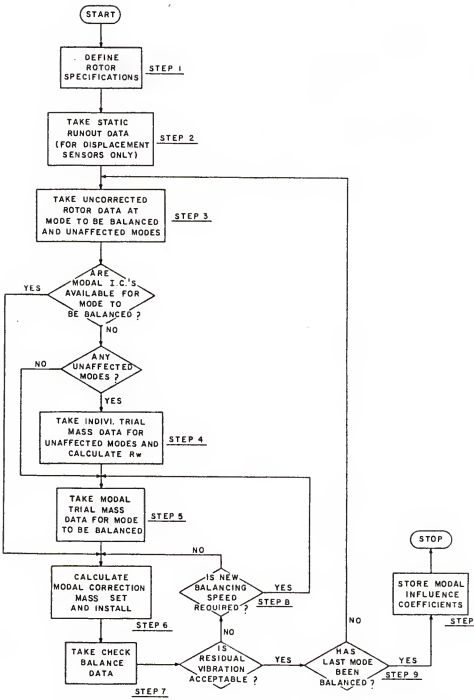



Figure 6.1 Flow chart of procedure for prototype Unified Balancing Approach system

changed by the operator during the balancing procedure, if necessary); previously measured modal influence coefficients for this specific rotor or class of rotors, if available; the number of weighted least-squares minimization iterations to perform if $n > m - 1$ or $N > 1$ (the convergence limit is set in the program to be 0.001); adjacent sensor/plane combinations for use in reciprocity calculations, if appropriate; sensor pairs to be used for observation of forward precession whirl, if appropriate; and any balancing specifications to be applied in steps seven and eight. This is the only step in this balancing procedure which requires engineering-level decisions to be made. The remainder of the steps may be handled by a moderately skilled operator. Thus, this balancing procedure is well suited for application to production balancing, where step one needs to be done only once.

2. The rotor is run at a very slow speed (to ensure that no dynamic response is present), and readings are taken from displacement measuring sensors only. These readings are referred to as static runout data and are due to rotor surface eccentricity (at the sensor target) and other sources of synchronous data, such as an initial bend in the rotor, which are not caused by rotor mass unbalance. In general, this static runout data is independent of the rotational speed of the

rotor. The true vibratory response of the rotor at other rotational speeds is found by subtracting the static runout from the measured vibration readings. If none of the sensors measure rotor displacement, this step is bypassed.

3. The undisturbed rotor (no trial masses installed) is run, and vibration readings are taken for the mode to be balanced and for all modes which are not to be affected by the balancing procedure, at corresponding specified speeds. If previously measured modal influence coefficients are available for the mode to be balanced, the balancing procedure is continued with step six. If this is not the case, but there are no unaffected modes (which is usually the case when balancing the first rotor mode), the balancing procedure is continued with step five. Otherwise, step four follows.
4. Individual trial masses are installed in each of the planes to be used for balancing the mode of interest (one at a time), and vibration readings are taken at the specified data speed for each of the unaffected modes. If reciprocity is used, only one balancing plane is used here. Either one or two trial masses are used for each balancing plane, as specified in step one. The computer program then calculates the complex modal trial mass ratios (the vector $\underset{\sim}{r}_w$).

5. A modal trial mass set is installed (which consists of a single mass, if there are no unaffected modes), and vibration readings are taken at the specified data speed for the mode to be balanced. A second modal trial mass set is installed and another set of vibration readings are taken, if so specified in step one. The computer program then calculates the modal influence coefficients for the mode being balanced, based on this modal trial mass data and the uncorrected rotor data from step three.
6. The computer calculates the modal correction mass set for the mode being balanced, based on the modal influence coefficients and the uncorrected rotor data from step three (modified to compensate for measurement error if two modal trial mass sets have been used for this mode). The operator then installs this modal correction mass set.
7. Another set of vibration readings is taken at the specified data speed for the mode being balanced to determine whether sufficient improvement of the rotor response has been achieved by this balancing procedure. The measured residual vibration is either displayed to the operator, for the operator to evaluate, or is compared to previously specified balancing criteria (from step one). If the measured residual vibration is determined to be acceptable, the balancing procedure

is continued with step nine; otherwise step eight follows.

8. If additional improvement in the measured residual vibration is required, and if it is determined that this improvement can be achieved using the same balancing speed (using the same modal influence coefficients), the balancing procedure is continued with step six. If, however, it is determined that a more sensitive balancing speed (nearer to the critical speed) is required, or that updated modal influence coefficients are needed (due to non-linearity in the rotor system), the balancing procedure is continued with step five. A balancing speed nearer to the relevant critical value may also be required to isolate the mode being balanced from other modes.

9. If one or more modes remain to be balanced, the mode just balanced is added to the list of unaffected modes; the next mode to be balanced is chosen (as specified in step one), and the balancing procedure continues with step three.

10. After the last mode has been balanced, the modal influence coefficients which have been measured may be stored for later use in rebalancing the same rotor or in balancing other rotors of the same type. This completes the balancing procedure for this rotor.

It should be noted that, when using measured modal influence coefficients to apply to a class of rotors, it may be necessary to measure the modal influence coefficients for several of these rotors and statistically average these values (as in reference [102]) to get a set of modal influence coefficients which can be applied with good results to a whole class of rotors.

## 6.4 Outline of a UBA Field Balancing System

Combined with modern microcomputer hardware, the Unified Balancing Approach (UBA) forms an operator-oriented balancing tool which is ideally suited to fill the current gap in flexible rotor balancing capability, particularly for use in the field.

This combination has been made possible by the recent developments in microprocessors and microcomputers. Currently, microcomputers are inexpensive, compact and relatively portable, and possess an impressively powerful computational ability. Thus, a flexible rotor balancing system is described in this section which combines the advantages of the UBA procedure with the power and portability of the modern microcomputer [115]. This system is ideally suited for field balancing applications (where portability is essential) and is also suitable for production and laboratory applications. The general architecture (hardware and software) of such a system is described below, along with the details of a specific system utilizing an MCS6502 microprocessor, being prototyped on an Apple II (Apple Computer Inc.) microcomputer.

There are, of course, many possible hardware configurations for a balancing system such as that referred to above. However, in most

cases, the same set of generic hardware components would be used. These components are listed below, along with their specific functions in the system design. A specific prototype system is also described.

The actual hardware configuration depends upon the intended use of this system. A minimum configuration (with no unnecessary peripheral devices) would be most desirable for field or production balancing applications where portability and/or limited operator flexibility (and minimum cost) are important. Such a minimum configuration would include:

1. A microprocessor to provide system control and perform all necessary calculations
2. Read-only memory (ROM) for storing the system software
3. Random-access memory (RAM) for storing balancing data and calculation results
4. Analog-to-digital converters (with analog multiplexors) for digitizing sensor data
5. Numeric keypad for input of trial mass information
6. Light-emitting diode (or similar) numeric display for output of balancing sequence instructions (by code) and correction mass information

Alternatively, the LED numeric display could be replaced with, or enhanced by, a small twenty-column printer, such as is commonly used for printing calculators. A minimum configuration system such as this could be assembled in a package no larger than a shoebox. Together with the necessary sensors, associated instrumentation, trial and correction masses (and/or balancing clay) and a few pads of paper, this system

would fit comfortably inside a standard briefcase. While a system of this type using currently avaiable hardware requires external power, the current development of CMOS devices will likely lead to a battery-powered system which will further enhance portability.

For laboratory, or similar, application where more flexibility and/or more exotic input/output (I/O) is desirable, additional peripheral devices may be added to the minimum configuration system. For example, the use of a standard terminal for I/O could be used for providing more detailed instructions for the user and report quality documentation of the balancing operations. In addition, a cathode ray tube (CRT) display could be used to provide graphical output (possibly, with hard-copy capabilities). The use of a large capacity data storage device (such as a floppy disk) would provide archiving capabilities and storage for additional programs. In this case, if the ROM listed earlier were replaced with RAM and the balancing program saved on disk, the system could be used as a conventional general computing system. This is, in fact, the form of the prototype system described below.

A prototype development system has been designed around the MCS6502 microprocessor [116]. The heart of the system is an Apple II microcomputer (Apple Computer Inc.). A mini-floppy disk is used for program and mass data storage. The microcomputer comes equipped with a full, typewriter-style keyboard. The primary output device is a CRT display for alpha-numerics and graphics. Hardcopies of program listings and output (including graphics) is accomplished with an eighty-column dot-matrix line printer. Sensor data are acquired by means of a sixteen-channel analog-to-digital converter (available on a single board).

This A/D converter system is actually composed of a single A/D converter and a sixteen-channel analog multiplexor.

The specific hardware design of an actual balancing system entails the consideration of several design objectives, whose relative importance is a function of the particular application. These design objectives include minimizing cost, minimizing size, providing portability (to be further enhanced with the use of battery-powered CMOS devices), and providing durability (primarily a function of packaging). These particular design objectives all favor the minimum configuration system. However, for designs in which increased operator flexibility and/or general computing capabilities are of primary importance, a system more akin to the development system described above would be appropriate.

The balancing system software could also be characterized as being composed of a number of individual components. These components include:

1. A main program which provides the overall system operational control and sequencing of balancing and software operations
2. A data acquisition routine for control of the A/D converter and multiplexor for digitizing the sensor data
3. A Discrete Fourier Transform (DFT) routine for extracting the magnitude and phase angle of the synchronous components of each set of sensor data
4. I/O routines for providing communication with and direction to the operator. The structure of these

routines (which could be designed specifically for each application) will determine the amount of flexibility permitted to the operator

5. Routines for performing the balancing calculations
6. Data archival routines (for more complex systems only)

The software design for a specific balancing system will involve merely the assembly of these components in an appropriate fashion, with modifications generally required for the main program and I/O routines only. Thus, for example, a very tightly structured procedure would be most desirable for a technician level operator in a production situation, while a much more loosely structured procedure (leaving more decisions to the operator) would be more desirable for a laboratory situation with direct engineering level supervision.

# CHAPTER 7
# EXPERIMENTAL VERIFICATION OF THE UNIFIED BALANCING APPROACH

In order to evaluate the true merits of any balancing method, it is necessary to apply it to actual hardware. To be credible, a balancing method must have, in addition to theoretical appeal, demonstrated practical appeal. That is, it is necessary to demonstrate both effectiveness and usability in an actual hardware application.

Consequently, the Unified Balancing Approach was experimentally verified in a test program involving a supercritical power transmission shaft test rig. In addition to application of the Unified Balancing Approach, these tests also involved comparisons with influence coefficient and modal balancing procedures.

The test rig was designed to simulate an advanced power transmission shaft for a heavy lift helicopter [83]. The principal component of the rig is a long, flexible shaft with an unsupported span of about 3.7 meters (12 feet). A test program was conducted using this test rig to compare the three balancing methods discussed above: modal balancing, influence coefficient balancing, and the Unified Balancing Approach (UBA). Influence coefficient balancing was performed first. The second series of tests used an enhanced modal balancing procedure. These enhancements included some of the data manipulation and least-squares minimization tools generally associated with influence

coefficient balancing. This test program culminated in the verification of a fully developed UBA procedure and computer program in the third set of tests. Details of the test rig and the test results follow.

## 7.1 Test Rig Description

The test rotor system [85] provided a drive system capable of speeds to 3,600 rpm, a pair of gearboxes each with approximately 5.7:1 speed ratio, and a high-speed test shaft, which could be driven at speeds up to 20,000 rpm, and would pass through several bending critical speeds in this speed range. The two gearboxes were arranged so that their high-speed and low-speed output shafts could be connected in a "four-square" closed loop arrangement into which a torque could be locked. In this way, power levels up to 2,500 hp could be simulated. The rig is shown in the sketch in Figure 7-1 (without the damper) for testing an unloaded shaft. For the purpose of the tests described herein, the test shaft was run in the unloaded configuration.

The test shaft was a hollow shaft 3.66 m (12 ft) long and 7.62 cm (3 in) in diameter with a 3.175 mm (0.125 in) wall thickness. The shaft had seven balancing rings and was supported at both ends by disk-type flexible couplings. One of the couplings was attached to a hard-mounted spindle, while the other was attached to a stub-shaft that was supported by a damper.

An undamped critical speed analysis of the test rig, including the damper shaft, was conducted. Figure 7-2 shows the predicted mode shapes and critical speeds for the first five modes. It should be noted that for all of these modes there is some significant amplitude at the

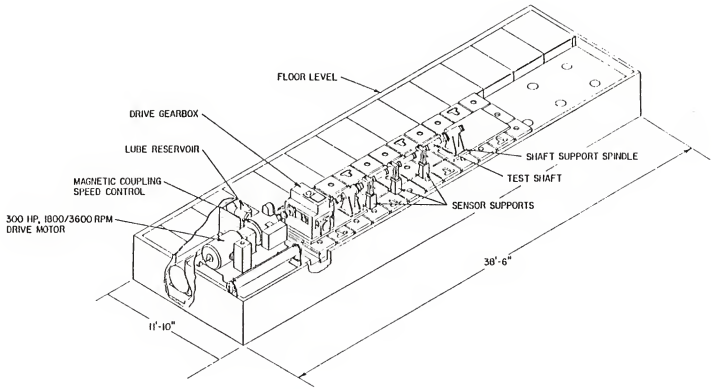



Figure 7-1 Drive train dynamics technology test rig configured for initial high-speed shaft system balancing demonstration

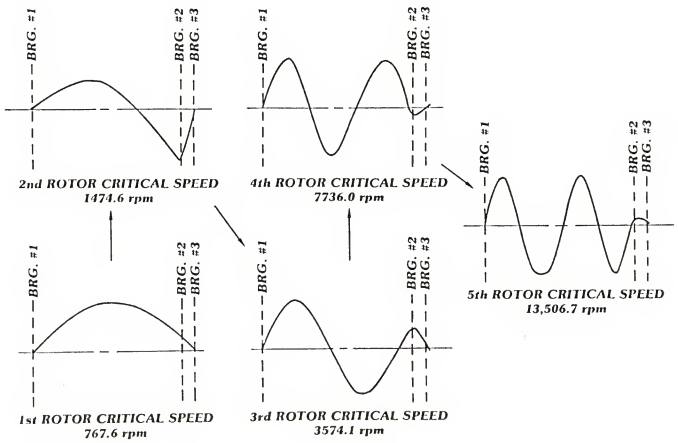



Figure 7-2 Mode shapes of the first five critical speeds

location of the damper, as represented by bearing number 2. This is important to achieve the necessary dissipation of vibration energy at the damper.

Next, a damped natural frequency and unbalance response analysis was conducted for the test rig in order to optimize the damping at each critical speed. The objective of the study was to find that combination of practical support properties (stiffness and damping) which maximized the rotordynamic stability of the test rig. The predicted damped critical speeds for the first and third through fifth modes were 853, 3,383, 7,500, and 13,200 rpm, respectively. The predicted damped mode shapes were essentially identical to the undamped mode shapes presented in Figure 7-2. The damped natural frequency analysis indicated that the second mode would be critically damped. As this proved to be the case, in keeping with the definition of mode in the attached glossary, the first, third and fourth critical speeds shall henceforth be referred to as the first, second and third modes (which occurred at about 940, 3,500, and 7,600 rpm, respectively).

Test shaft vibration was measured by means of eddy current type displacement probes located adjacent to the balancing rings and at the ends of the damper. A sketch indicating the location and reference number of each of these displacement probes is presented in Figure 7-3. The probe or sensor numbers referred to below correspond to the reference numbers in Figure 7-3. The sketch in Figure 7-4 illustrates the relative locations of the displacement probes and the phase reference sensor and the orientation of the correction mass holes.

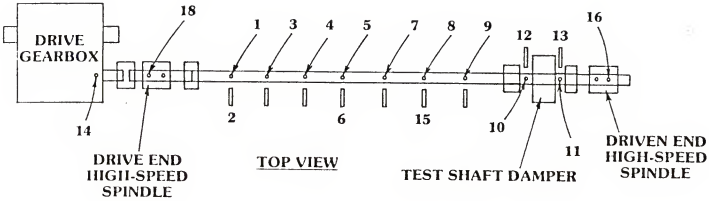



Figure 7-3 Sketch showing displacement probe locations as referred to by probe numbers

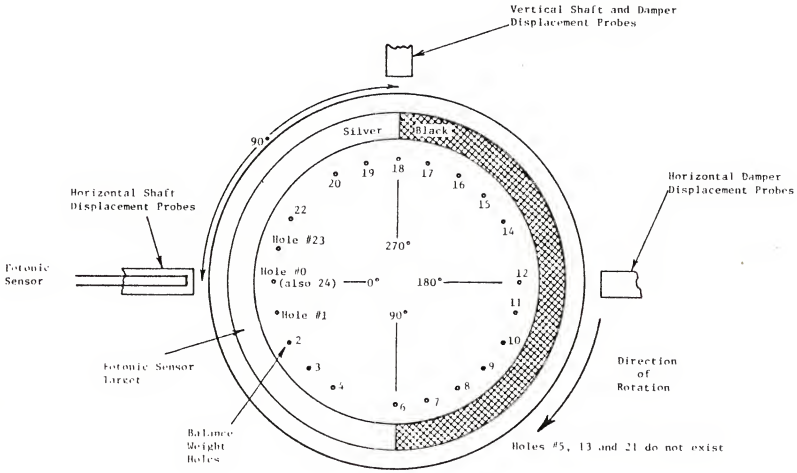



Figure 7-4 Sketch showing phase angle reference of displacement probes, fotonic sensor and correction weight holes with test shaft in zero degree position

During the tests, measurement of synchronous test shaft vibration was extracted from the displacement probe signals through the use of a minicomputer controlled data acquisition system. The data acquisition system included a twenty-channel scanner, tracking filter, phase meter, digital voltmeter and tachometer. The minicomputer served the dual purpose of controlling the data acquisition system and analyzing the acquired data. Sophisticated influence coefficient balancing software installed on the minicomputer was used during the test. Two damper types were used at different stages of the test program. A squeeze-film damper was used during the influence coefficient and modal balancing tests; an elastomer damper was used during the UBA tests. A split sketch showing the two damper designs is presented in Figure 7-5. The response of the system with the elastomer damper did not differ from that with the squeeze-film damper except for a slight change in the critical speeds. The shapes of the resonant peaks and the sensitivity of the test rig to unbalance was not noticeably affected by the change in damper configuration. Thus, the type of damper is not a significant factor in the comparisons of balancing test results. Additional details of the design and analysis of the test rig, as well as a more complete description of the hardware, can be found in references [83], [84], and [85].

## 7.2 Influence Coefficient Balancing Tests

The first set of balancing tests involved the use of an influence coefficient balancing procedure. An on-line computer program was used to handle data acquisition and balancing calculations [60]. The

ELASTOMER DAMPER/SQUEEZE-FILM DAMPER SCHEMATIC

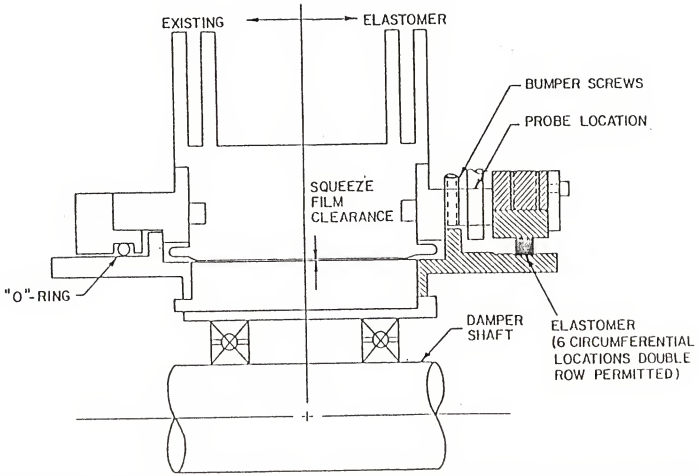



Figure 7-5 Side-by-side sketch of squeeze-film and elastomer dampers designed to suppress vibrations in a supercritical power transmission shaft

balancing calculation portion of the program was based on the work of Lund [55].

This series of tests is described in detail in references [83] and [117]. Briefly, the test shaft was balanced for the first and third critical speeds with no difficulty. However, the first attempt at balancing for the fourth critical speed resulted in substantial degradation of the balance condition for the first and third critical speeds. After rebalancing at the first and third critical speeds, the test shaft was successfully balanced through the fourth critical speed to over 10,000 rpm. A summary of the balancing runs is presented in Table 7-1, and a plot of the test rig response is presented in Figure 7-6.

## 7.3 Enhanced Modal Balancing Tests

An enhanced modal balancing procedure was used for the second set of tests. The basic procedure is based on the work of Bishop and Parkinson [35]. The on-line computer program described in the previous section was also used for data acquisition and some of the balancing calculations for these tests. This program enhanced the standard modal balancing procedure. In particular, extra trial mass runs compensated for measurement errors, and a least-squares solution improved the correction mass calculations for multiple sensor locations. These features are not generally associated with modal balancing. In fact, these tests represented the first step in the development of the Unified Balancing Approach.

This series of tests is also described in detail in references [83] and [117]. The test shaft was successfully balanced through the

Table 7-1 Balancing masses and vibration data for influence coefficient balancing tests

| | | | Initial Vibration | | Correction Masses | | | | | | Subsequent Vibration | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | Plane 2 | | Plane 4 | | Plane 6 | | | | |
| Run No. | Mode No. | Data Speed (rpm) | Amplitude [mm(mils)P-P] | Angle (Degrees) | Mass (grams) | Angle (degrees) | Mass (grams) | Angle (degrees) | Mass (grams) | Angle (degrees) | Amplitude [mm(mils)P-P] | Angle (degrees) | Probe No. 1 |
| 1 | 1 | 901 | 2.15(84.6) | -169 | - | - | 5.787 | - 63.2 | - | - | 1.33(52.3) | 175 | 5 |
| 2 | 1 | 901 | 1.33(52.3) | 175 | - | - | 4.555 | - 79.9 | - | - | 0.33(13.5) | 132 | 5 |
| 3 | 1 | 901 | 0.34(13.5) | 152 | - | - | 1.070 | -120.0 | - | - | 0.14 (5.7) | 73 | 5 |
| 4 | 3 | 3200 | 0.22 (8.5) | -151 | 1.632 | -82.3 | 1.571 | 102.7 | 0.627 | -113.2 | 0.07 (2.7) | -148 | 3 |
| 5 | 4 | 6800 | 1.34(52.9) | - 58 | 4.45 | -66.1 | 6.04 | 119.4 | 3.77 | - 49.7 | 0.27(10.7)* | - 32* | 5 |
| 6 | 3 | 3200 | 0.16 (6.2) | - 33 | 0.273 | 90.0 | 0.26 | 0.0 | 1.27 | -127.9 | 0.03 (1.3) | 140 | 3 |
| 7 | 3 | 3200 | 0.03 (1.3) | 140 | 0.45 | -75.7 | 0.63 | 133.0 | 0.65 | - 54.6 | ~0.01† | † | 3 |
| 8 | 4 | 6800 | 0.27(10.7) | - 32 | 1.30 | -39.5 | 1.16 | 132.3 | 0.44 | 5.5 | 0.05 (1.9) | 17 | 5 |
| 9 | 4 | 6800 | 0.13 (5.1) | 149 | 0.31 | 60.0 | 1.02 | 129.9 | 0.26 | 38.6 | 0.05 (2.0) | 151 | 3 |
| 10 | 4 | 6800 | 0.05 (2.0) | 151 | 0.35 | 150.0 | 0.17 | 105.0 | 0.14 | 150.0 | ~0.03† | † | 3 |
| 11 | 3 | 3200 | 0.11 (4.2) | -166 | 0.74 | 131.9 | 0.48 | - 6.7 | 0.38 | -150.9 | ~0.04† | † | 8 |

*This data was taken after run 7, the first time that 6800 rpm could be reached again.

†Exact data not available.

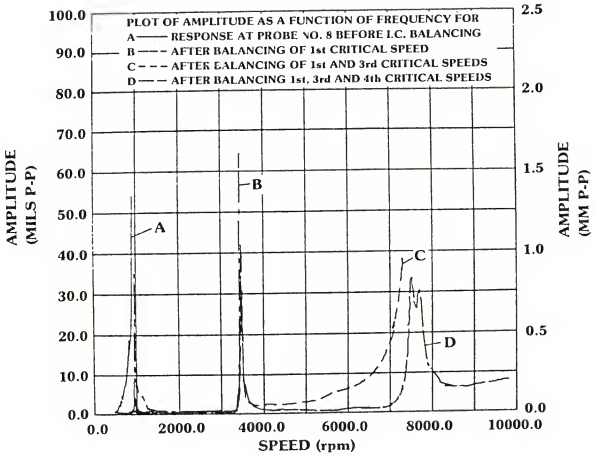



Figure 7-6 Respone at probe no. 8 during influence coefficient balancing

first, third and fourth critical speeds. The correction for the fourth critical speed did not have a substantially detrimental effect on the response at either the first or third critical speed. The mode shapes were observed to be nearly, but not exactly, planar. The test rig was again run to over 10,000 rpm. A summary of the balancing runs is presented in Table 7-2 and a plot of the test rig response is presented in Figure 7-7.

## 7.4 Unified Balancing Approach Tests

The third set of balancing tests verified the Unified Balancing Approach. These tests used an on-line computer program similar to that used for the first two sets of balancing tests. However, the software was reconfigured for the procedure and the computations of the UBA [113]. Details of the theory and procedure for the UBA are presented in Chapter 6. In general, these tests were successful, and the superiority of the UBA (at least for this test rig) was evident. Details of these tests follow [118].

The initial condition of the rotor was such that the first critical speed could not be negotiated. (The test rig could not be operated safely above 900 rpm.) The initial condition was comparable to the unbalance condition that existed prior to each of the earlier test programs. Since a large correction mass was anticipated for the first critical speed, the correction was spread to three balance planes (2,4, and 6) by installing 3 trial masses simultaneously. (Note: the procedure for balancing the first mode was identical for the three balancing methods, as applied in these tests.)

Table 7-2 Balancing masses and vibration data for enhanced modal balancing tests

| | | | Initial Vibration | | Correction Masses | | | | | | Subsequent Vibration | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | Plane 2 | | Plane 4 | | Plane 6 | | | | |
| Run No. | Mode No. | Data Speed (rpm) | Amplitude [mm(mils)P-P] | Angle (Degrees) | Mass (grams) | Angle (degrees) | Mass (grams) | Angle (degrees) | Mass (grams) | Angle (degrees) | Amplitude [mm(mils)P-P] | Angle (degrees) | Probe No. 1 |
| 1 | 1 | 901 | 0.89(35.1) | 165 | - | - | 3.51 | -97.0 | - | - | 0.03(1.0) | 22 | 5 |
| 2 | 2 | 3200 | 1.17(46.1) | -16 | 5.52 | 160.7 | - | - | 5.52 | -19.3 | 0.17(6.5) | 121 | 3 |
| 3 | 2 | 3200 | 0.17 (6.5) | 121 | 0.32 | -59.7 | - | - | 0.32 | 120.3 | 0.09(3.5) | 125 | 3 |
| 4 | 3 | 6400 | 1.20(47.4) | -53 | 8.50 | -57.5 | 11.89 | 122.5 | 8.50 | -57.5 | 0.07(2.9) | 76 | 5 |
| 5 | 1 | 901 | 0.21 (8.1) | 63 | - | - | 0.80 | 160.9 | - | - | † ~0.6 | † | 5 |

†Exact data not available.

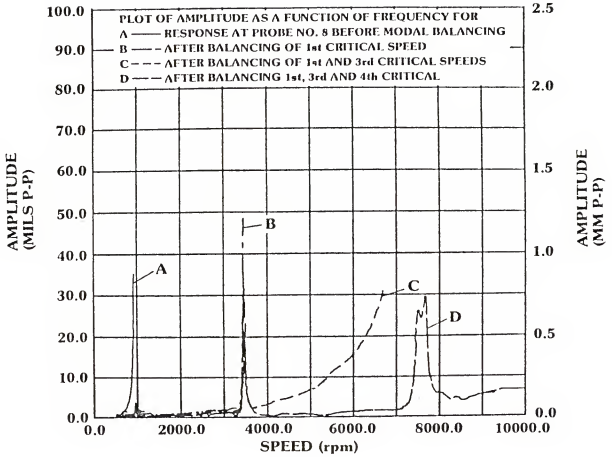



Figure 7-7 Response at probe no. 8 during enhanced modal balancing

Uncorrected rotor and modal trial mass (equal masses in planes 2, 4, and 6) data were taken for the first mode at 890 rpm. A correction mass of about 2.3 grams at about 330 degrees in each of planes 2, 4, and 6 was calculated. These masses were fabricated and installed.

The test rig was run and check balance data were taken at 893 rpm (the closest steady speed to 890 rpm). The net synchronous vibration of the rotor was reduced by more than 40 percent, but the first critical speed was not yet negotiable. A trim correction of about 1.3 grams at about 270 degrees in each of planes 2, 4, and 6 was calculated. These correction masses were fabricated and installed. The response of the rotor at the first critical speed was reduced tremendously and the first critical speed was negotiated easily. Check balance data which indicated the presence of measurable residual vibration were then taken. Based on these data, a set of trim corrections of about 0.6 grams at 165 degrees in each of planes 2, 4, and 6 was calculated. These correction masses were also fabricated and installed.

With the third set of correction masses installed, the test rig was run and check balance data were taken at 889 rpm. These data indicated a substantial improvement (better than a 95 percent reduction of the synchronous vibration as compared to the unbalanced condition). The vibration at the first critical speed was well controlled and the test rig was capable of safe operation up to about 7,800 rpm (the fourth critical speed was increased to about 8,000 rpm).

Clearly, the second critical speed (about 2,500 rpm), while no longer critically damped (with the squeeze-film damper it was critically damped), was heavily damped and nearly unresponsive to

unbalance. The third critical speed, on the other hand, although fairly heavily damped, did evidence some response to unbalance. Thus, the balancing procedure was continued with the third critical speed (treated as the second mode). Again, the on-line computer program was used to take data and perform the Unified Balancing Approach calculations. During the individual trial mass portion of the balancing procedure, planes 2 and 4 were mistakenly used, instead of planes 2 and 6 as originally intended. Thus, the modal trial mass set that was calculated by the program was not longitudinally symmetric, as it was expected to be. That is, it was antiipated that the modal trial mass set for planes 2 and 6 would be composed of a mass in plane 2 approximately equal in magnitude and opposite in direction to that in plane 6. However, the use of planes 2 and 4 resulted in a modal trial mass set composed of a mass in plane 2 approximately 50 percent larger in magnitude than that in plane 4, with the angle of the two masses differing by about 170 degrees. Clearly, the combination of planes 2 and 4 was not the optimum plane combination to be used for balancing the second mode (third critical speed). However, it was anticipated that the use of such non-ideal planes might provide an interesting test of the Unified Balancing Approach. Consequently, the modal trial mass set composed of planes 2 and 4 was used for balancing the second mode (third critical speed). This test was very successful, as the modal correction mass set that was calculated and installed resulted in a substantial improvement of the response at the third critical speed (better than a 70 percent reduction) while having virtually no effect on the response at the first critical speed. A trim correction was

calculated, but it was not installed because it was quite small and the response at the third critical speed was already well controlled.

The test rig was not run again until the following day. Without adding or removing any trial or correction masses, the test rig was run up to a speed of about 7,800 rpm. Although the synchronous response at the second and third critical speeds was virtually unchanged from the previous run, the response at the first critical speed was much worse. Apparently something in the shaft or test rig had changed spontaneously to alter the amount of distribution of unbalance or the amount of damping. It is possible that the change in the test rig response was due to a change in the shape or support (that is, a shift) of the test shaft or a set in the elastomer damper due to the test shaft weight. In any case, the balancing procedure was continued with a trim balance at the first critical speed.

The computer program was again used to apply the Unified Balancing Approach. This time a data speed of 911 rpm was used for balancing the first mode. Since this speed is closer to the first critical speed than the balancing speed of 890 rpm used during earlier runs, more sensitive balancing data could be acquired. A balancing run for the first critical speed was conducted at this balancing speed. Also, the trial mass (and subsequently the correction mass) was composed of just a single mass in plane 4 rather than distributed between planes 2, 4, and 6. The uncorrected rotor data taken during this balancing run indicated that the synchronous response of the test rig near the first critical speed was more than double that at the end of the previous run. For the first mode, a correction mass of about 1.7 grams

at about 44 degrees in plane 4 was calculated and installed. The improvement in the response at the first critical speed was dramatic. A set of check balance data for the first mode was taken and a reduction in synchronous response as measured by the various probes of between 75 and 93 percent was indicated. This correction mass had virtually no effect on the response at the third critical speed (second mode). This absence of effect was not surprising in light of the fact that the second mode had a node very near to the location of plane 4.

It was not necessary at this point to do any more balancing at any of the first three critical speeds. Before continuing balancing with the fourth critical speed (treated as the third mode) several test runs were conducted and the synchronous response of the test rig was observed, particularly around the first critical speed. The response was repeatable except around the first critical speed where some substantial variations in response were observed. These variations were usually small (within 25 percent of the "balanced" response). However, at one point the rotor data at the first critical speed increased by more than a factor of two, only to return to its "balanced" response on the following run. There appeared to be a shift in the test shaft followed by a shift back. After this run the test rig response became more repeatable.

Uncorrected rotor data were taken for the first three modes (first, third, and fourth critical speeds), followed by a series of individual trial mass runs in planes 2, 4, and 6. Data for these trial mass runs were taken in the vicinity of the first and third critical speeds to permit the calculation of a modal trial mass set for the third

mode (fourth critical speed). The modal trial mass set calculated for the third mode consisted of a mass of 1.0 gram at zero degrees in plane 2 (reference plane), a mass of about 1.4 grams at about 191 degrees in plane 4, and a mass of about 1.16 grams at about 3.6 degrees in plane 6. Note that this modal trial mass set was almost, but not quite, planar. A balancing run was conducted for the third mode (fourth critical speed) using this modal trial mass set. The modal trial mass set had no noticeable effect on the response at any of the first three critical speeds (neglecting the run-to-run variations at the first critical speed). A modal correction mass set was calculated for the third mode. This set consisted of a mass of about 1.3 grams at 30 degrees in plane 2, 1.8 grams at 221 degrees in plane 4, and 1.5 grams at 34 degrees in plane 6. These masses were fabricated and installed. When the test rig was run, a remarkable improvement was witnessed at the fourth critical speed. The fourth critical speed was negotiated easily; there was no noticeable effect on the response at either the second or third critical speed, and any change in the response at the first critical speed could easily be attributed to run-to-run variation in that response. (The observation of the operators was that the response at the first critical speed after this balancing run was not noticeably different from the response immediately prior to installing the correction masses). A set of check balance data taken at the balancing speed for the fourth critical speed (7,620 rpm) indicated a reduction in synchronous rotor response of better than 90 percent. A trim balance mass set was calculated, but the masses were impractically small (0.1 gram), and the rotor response was so good that the trim was

not installed. This was clearly the best balanced condition that had ever been achieved with this test rig.

> Recalling that at least two days and six balance runs were required to balance just the fourth critical speed during the initial influence coefficient balancing procedure, the one run balance accomplished using the Unified Balancing Approach impressively demonstrated the benefits in balancing efficiency to be gained from the UBA.

The modal mass set ratios are listed in Table 7-3 and are illustrated schematically in Figure 7-8 and 7-9 (only the plane 4 mass is shown for the first mode). The axial distribution of the trial masses is shown in Figure 7-8; the angular distribution is shown in Figure 7-9. Note that the sets are almost, but not quite planar. A tabulation of the correction masses that were added and the resultant change in vibration at key speeds during this series of tests is presented in Table 7-4. The sequence of balancing operations is illustrated in Figure 7-10 by a series of vector plots, pairs of which show the correction mass added in the reference plane for each mode and the resulting change in vibration for the corresponding mode. A set of typical rotor response plots is presented in Figure 7-11.

After the completion of these tests, the sets of individual influence coefficients measured near the first and third critical speeds (individual influence coefficients were not measured near the fourth critical speed) were examined for reciprocity effects, as discussed in

Table 7-3 Modal mass set ratios

| Mode No. | Plane 2 | | Plane 4 | | Plane 6 | |
|---|---|---|---|---|---|---|
| | mass (grams) | angle (degrees) | mass (grams) | angle (degrees) | mass (grams) | angle (degrees) |
| 1 | --- | --- | 1.0* | 0 | --- | --- |
| 2 | 1.0 | 0 | 0.666 | -170.6 | --- | --- |
| 3 | 1.0 | 0 | 1.393 | -169.4 | 1.165 | 3.6 |

*The plane with the unit mass is the reference plane

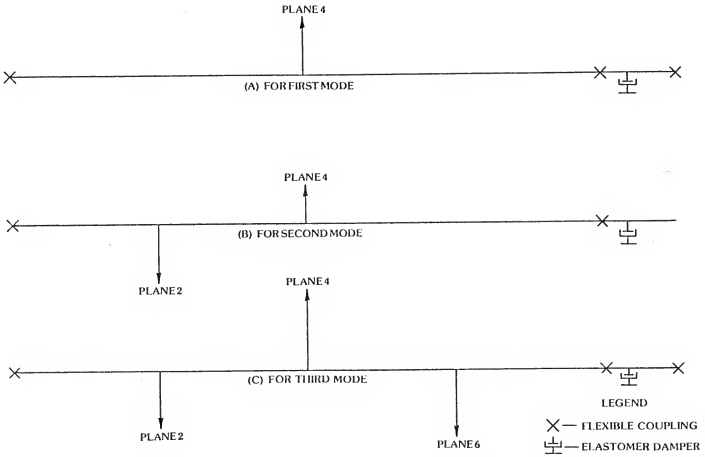



Figure 7-8 Longitudinal distribution of modal mass sets

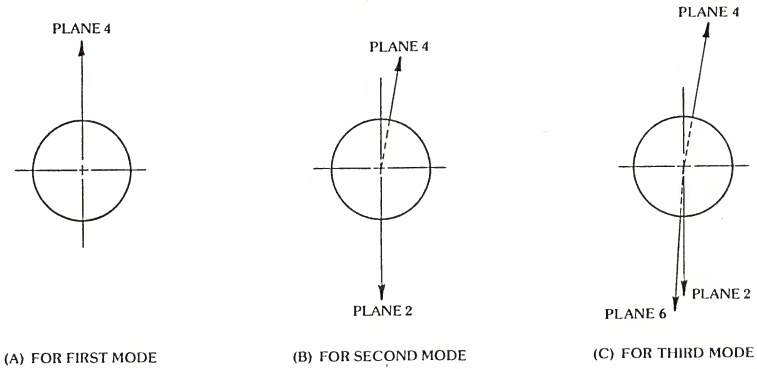



Figure 7-9 Angular distribution of modal mass sets

Table 7-4 Balancing masses and vibration data for unified balancing approach tests

| | | Initial Vibration | | | Correction Masses | | | | | | Subsequent Vibration | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Data | | | Plane 2 | | Plane 4 | | Plane 6 | | | | |
| Fig. 8 Ref. | Mode No. | Speed (rpm) | Amplitude [mm(mils)P-P] | Angle (degrees) | Mass (grams) | Angle (degrees) | Mass (grams) | Angle (degrees) | Mass (grams) | Angle (degrees) | Amplitude [mm(mils)P-P] | Angle (degrees) | Probe No. |
| (a) | 1 | 890 | 1.88(74.19) | 299.3 | - | - | 2.305* | - 32.6 | - | - | 1.10(43.40) | - 44.2 | 5 |
| (b) | 1 | 890 | 1.10(43.40) | - 44.2 | - | - | 1.344* | - 90.0 | - | - | 0.45(17.67) | -141.2 | 5 |
| (c) | 1 | 890 | 0.45(17.67) | -141.2 | - | - | 0.591* | 165 | - | - | 0.06 (2.30) | 24.8 | 5 |
| (d) | 2 | 3820 | 0.32(12.50) | -149 | 1.263* | 177.6 | 0.841 | 6.9 | - | - | 0.10 (4.08) | -116.2 | 3 |
| (e) | 1 | 911 | 1.11(43.83) | 60.6 | - | - | 1.692* | 43.5 | - | - | 0.07 (2.91) | 171.3 | 5 |
| (f) | 3 | 7620 | 0.90(35.34) | -53.3 | 1.283* | 30 | 1.787 | -139 | 1.495 | 34 | 0.07 (2.94) | 106.6 | 5 |

*Denotes reference plane.

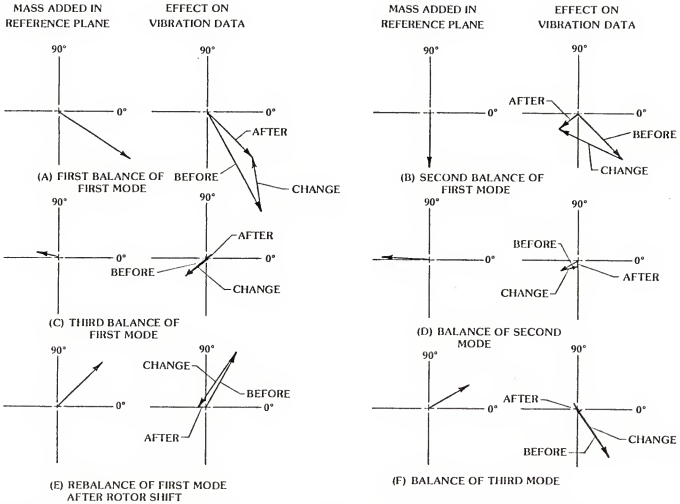



Figure 7-10 Effect of balance masses on vibration data for the Unified Balancing Approach tests

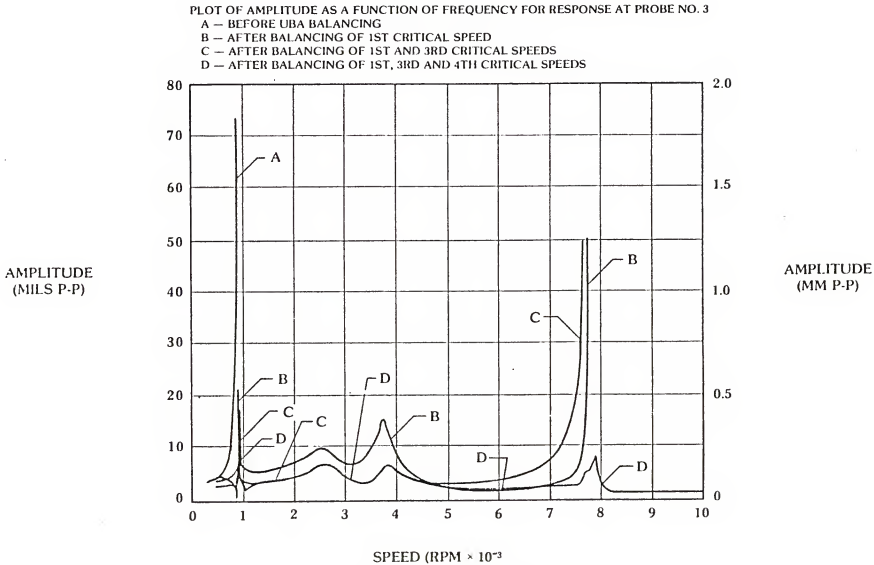



Figure 7-11 Response at probe no. 8 during use of Unified Balancing Approach

Chapter 6. In general, the principal of reciprocity appeared to apply within a reasonable tolerance (a few percent). However, no actual balancing tests were performed using the principle of reciprocity to reduce the number of individual trial mass runs, as proposed in Chapter 6.

## 7.5 Discussion of Results

The results of the three test programs are compared in Table 7-5. Since the three procedures were essentially the same for balancing the first mode (first critical speed), it is not surprising that the corresponding results were very similar. However, the results for balancing of the second and third modes (third and fourth critical speeds) indicate that the Unified Balancing Approach is superior to the other two methods in terms of both efficiency and effectiveness.

Specifically, the UBA required many fewer total runs and highest-speed runs than did influence coefficient balancing. Also, the UBA did not require any rebalancing at the lower critical speeds and obtained good sensitivity near the fourth critical speed with no difficulty, unlike influence coefficient balancing. In addition, the final balance condition obtained using the UBA was substantially better than that obtained using influence coefficient balancing.

While the results obtained using the Unified Balancing Approach were superior to those obtained using the enhanced modal procedure, the difference was not as dramatic as the comparison with influence coefficient balancing. However, this does not indicate that modal balancing is superior to influence coefficient balancing, since the enhanced modal procedure incorporated some of the features of influence

Table 7-5 Number of balancing runs

| Mode No. | Balancing Tries | Total Runs | Data Sets | Highest Speed Runs | Residual Vibration mm (mils) p - p |
|---|---|---|---|---|---|
| 1 IC* | 3 | 5 | 5 | 5 | 1.0 (39) |
| 1 EM** | 2 | 4 | 4 | 4 | 0.2 (8) |
| 1UBA*** | 3 | 5 | 5 | 5 | 0.5 (20) |
| 2 IC | 4 | 10 | 20 | 10 | 1.2 (46) |
| 2 EM | 2 | 8 | 8 | 4 | 0.8 (32) |
| 2 UBA | 1 | 7 | 7 | 3 | 0.2 (7) |
| 3 IC | 4 | 10 | 30 | 10 | 1.6 (62) |
| 3 EM | 1 | 9 | 9 | 3 | 1.2 (49) |
| 3 UBA | 1 | 9 | 9 | 3 | 0.2 (9) |

*During influence coefficient balancing

**During enhanced modal balancing

***During use of the Unified Balancing Approach

coefficient balancing and was, in fact, the first step in the development of the UBA. In general, the final balance condition was better using the UBA than the enhanced modal balancing. Also, the UBA was more straightforward and less intuitive than the enhanced modal procedure.

The abstracts and summary from references [83, 84, 85] and [118] may be found in Appendix A.

# CHAPTER 8
# SUMMARY AND CONCLUSIONS

Rotating machinery accounts for the vast majority of modern technological hardware. While the dynamic behavior of such machinery has always been important, the current emphasis on higher efficiency has resulted in the design of machines which are increasingly susceptible to vibratory phenomena. In particular, reduction in weight, coupled with increased operating speeds, have substantially increased the flexibility of rotating machinery. Thus, many types of machines are currently being designed to operate supercritically, and this trend is likely to continue. Consequently, reduction of mass unbalance in rotating machinery is often essential to ensure safe operation and reasonable life for such machines. This is particularly true for machines with flexible rotors. Currently, virtually all methods used for balancing flexible rotors fall into one of two classifications: modal balancing and influence coefficeint balancing. Each of these methods has specific advantages and disadvantages. These disadvantages have been of sufficient importance to preclude the general acceptance or widespread use of any of these balancing methods. Consequently, an original balancing procedure has been proposed and outlined herein (Unified Balancing Approach) which has been designed to incorporate the advantages of both modal and influence coefficient balancing while eliminating their disadvantages. Some of the advantages of the Unified Balancing Approach (UBA) are as follows:

1. No prior knowledge of the modes of vibration of the rotor to be balanced is required. The necessary information is determined empirically in the course of balancing.
2. As with modal balancing, the modes can be balanced individually, while not affecting the balance of other modes that have already been balanced.
3. In particular cases where the lower modes are more lightly damped than the higher modes, the use of modal trial mass sets allows the application of substantial trial masses at the insensitive modes, while not aggravating the response of the lower modes. It should be noted that individual trial mass runs that are required for the Unified Balancing Approach always involve modes that have already been balanced. Therefore, the vibration readings for these individual trial mass runs can always be taken at speeds close to the critical speeds so that small trial masses can be used with substantial effect, while the rotor is not prevented from going through lower critical speeds.
4. Using the Unified Balancing Approach, a small number of runs is required at the highest speeds of the rotor than would be required for influence coefficient balancing with individual trial mass runs. For rotors which obey the principle of reciprocity, and for which closely adjacent planes and sensors are available,

the total number of individual trial mass runs can also be substantially reduced.

5. In many cases, the balancing of a particular mode begins well below the corresponding critical speed, and the response at the balancing speed is reduced to the point where no additional improvement can be made (due to limited vibration pickup sensitivity), but the critical speed can still not be negotiated. Only a single trial mass run is required at a higher balancing speed using the Unified Balancing Approach. In contrast, using influence coefficient balancing, a complete new set of trial mass runs is required at the higher speed. Thus, use of the Unified Balancing Approach results in a substantial reduction in the number of balancing runs.
6. Production balancing of a series of identical rotors can be greatly simplified through the use of empirically determined and statistically averaged standard modal influence coefficients. In this way, the trial mass runs become few, or even nonexistent, for a production balancing situation.

In this study, the Unified Balancing Approach has been compared experimentally to influence coefficient and modal balancing. The superiority of the UBA has been demonstrated in terms of efficiency, effectiveness, and ease of application.

## GLOSSARY OF TERMS
## FOR THE UNIFIED BALANCING APPROACH

BALANCE MASS PLANE (Balance Plane)--Axial position on rotor of location for making balance mass corrections.

CHECK BALANCE--Data taken at the balancing speed after the addition of a modal correction mass set. If the improvement in the balance is not sufficient, new modal influence coefficients and an additional modal correction mass set may be calculated using the check balance data.

INDIVIDUAL TRIAL MASS--A single trial mass used to generate influence coefficients for individual balance mass planes; used for calculating modal trial mass set ratios.

INFLUENCE COEFFICIENTS--Complex values representing the effect of the addition of the trial masses on the rotor responses.

MODAL CORRECTION MASS SET--A group of correction masses used to balance a particular mode while having no effect on other particular modes. The number of masses (and balance planes) in this set is greater than or equal to the mode number of the mode being balanced. The magnitude ratios and relative angles of the masses in the set are the same as for the modal trial mass set used to excite the same mode, but the actual magnitudes and

angles are calculated using rotor response data from the modal trial mass runs.

MODAL INFLUENCE COEFFICIENTS--Complex values representing the effect of the addition of the modal trial mass sets on the rotor response.

MODAL TRIAL MASS SET--A group of trial masses used to excite a particular mode while having no effect on other particular modes. The number of masses (and balance planes) in this set is greater than or equal to the mode number of mode being excited. The magnitude ratios and relative angles of the masses in the set are calculated using rotor response data from individual mass runs, but an arbitrary complex multiplier can be applied to the set.

MODE--Shape of responses of rotor-bearing system at a critical speed.

MODE NUMBER--Generally, the order in which critical speeds occur when the speed of the rotor-bearing system is increased. For generally rigid bearings, this number refers to the number of anti-nodes in the mode shape. For generally flexible bearings with distinctly separate rigid body modes, this number will be one for the first rigid body mode and one greater than the number of rigid body modes (not including additional modes due to bearing anisotropy) for the first flexural mode. When balancing the $n^{th}$ mode, there are n-1 modes with lower critical speeds (and possibly some modes with higher critical speeds), at

which the vibration level must not be aggravated. Critically damped modes are not counted.

TRIAL MASS--A single mass or group of masses of known magnitude installed in a rotor at a specific balance plane and angle, to generate data used for calculating influence coefficients of the rotor-bearing system for this balance mass location (or combination) at speeds at which data were taken.

TWO TRIAL MASSES PER PLANE (Alternatively referred to as 0° and 180° trial masses--Trial mass data taken with a trial mass (or set of masses) located at the specified trial mass angle (or angles) and then moved to 180 degrees from that position (or those positions). The additional data is used to calculate the measurement error inherent in the test data and to correct for that error.

UNCORRECTED ROTOR--Rotor condition at the beginning of a balancing step, with no trial or correction masses installed.

# APPENDIX A

This appendix includes the abstracts (or summaries) of four references which provide more detailed information on the test rig used in this study.

[83] Darlow, M., and Smalley, A., Design and Test of a Scale Model Test Rig for Supercritical Power Transmission Shafting, MTI Technical Report No. 78TR41, NASA Contractor Report CR-3155, prepared for NASA-Lewis Research Center, under Contract No. NAS3-16824, (September) 1978.

Summary

To demonstrate the feasibility of supercritical power transmission shafting, a test rig has been designed, built, and successfully operated. Shafts of up to 6.1 meters (20 ft) in length can be accommodated, and controlled speeds to 20,000 rpm can be achieved; a controlled torque of up to 900 N-m (8000 in.-lb) can be imposed on the test shaft; extensive vibration instrumentation is available and good display of vibration signals is possible; a computerized data acquisition system allows on-line balancing and vibration monitoring; the facility and test vehicle provide a very sensitive tool for technology investigation associated with supercritical shafting.

Initial tests of a 3.66 meter (12 ft) long, 7.62 cm (3 in.) diameter, thin-walled shaft demonstrated, as expected, that balancing of a hard-mounted shaft without damping was difficult, and that once the first critical speed was negotiated, sustained subsynchronous vibrations at the frequency of the first critical speed dominated the dynamic motion of the shaft and led to an intolerable operating condition. A self-contained squeeze-film damper was designed and built which successfully controlled subsynchronous and synchronous vibrations so that balancing through four bending critical speeds was possible.

Tests under torque revealed that the shaft can undergo temporary deterioration in its state of balance due, apparently, to change in orientation of mass eccentricity vectors which are self-canceling under zero torque conditions. Investigations of nonsynchronous shaft vibration using a real-time analyzer revealed significant forced vibrations at bearing cage rotational frequency and at twice rotational speed, and also showed that for operation beyond 12 times the first critical speed, subsynchronous vibration at the frequency corresponding to the first critical speed can be observed even with a well-designed damper.

[84] Darlow, M. S., and Smalley, A. J., "Design and Test of a Squeeze-Film Damper for a Flexible Power Transmission Shaft," included in the ASME publication Topics in Fluid Film Bearing and Rotor Bearing System Design and Optimization, pp. 43-51, (April) 1978.

Abstract

For a flexible shaft designed to pass through a number of bending critical speeds, a squeeze-film damper has been designed and tested. The damper properties were selected to provide control of all critical speeds, while meeting additional constraints of high power transmission requirements and damper simplicity. The damper was fabricated and installed and its ability to control flexible shaft vibrations was demonstrated by the comparison of vibration amplitudes both with and without the damper.

[85] Darlow, M. S., Smalley, A. J., and Fleming, D. P., "Demonstration of a Supercritical Power Tranmission Shaft." published in the Proceedings of the National Conference on Power Transmission, Chicago, Illinois, (November 7-9) 1978.

Abstract

Operation of a very flexible power transmission shaft has been demonstrated in a laboratory environment through four bending critical speeds. Supercritical shafting has long been considered for use in transmitting power, particularly in aircraft applications where component size and weight is a principal design factor. Historically, the successful application of supercritical power transmission shafting has been limited by the past difficulty in producing well-balanced, sufficiently damped shafts which could operate safely and consistently. The design, analysis and successful operation of such a shaft is described in this paper. This shaft was fabricated from an aluminum tube 12 feet long and 3 inches in diameter with a 1/8 inch wall thickness. The shaft was supported at one end by a squeeze-film damper which was designed to provide control of all critical speeds while meeting additional constraints of high power transmission requirements and damper simplicity. This shaft was successfully balanced and operated through four bending critical speeds. The shaft was run in both torsionally loaded and unloaded configurations. When running under load, the shaft transmitted up to 1,000 hp at 12,000 rpm. The controllable and repeatable operation of a supercritical power transmission shaft was clearly demonstrated.

[118] Darlow, M. S., Smalley, A. J., and Parkinson, A. G., "Demonstration of a Unified Approach to the Balancing of Flexible Rotors," ASME Transactions, Journal of Engineering for Power, ASME Paper No. 80-GT-87, 1980.

Abstract

A flexible rotor balancing procedure, which incorporates the advantages and eliminates the disadvantages of the modal and influence coefficient procedures, has been developed and implemented. This new procedure, referred to as the Unified Balancing Approach, has been demonstrated on a supercritical power transmission shaft test rig. The test rig was successfully balanced through four flexural critical speeds with a substantial reduction in effort as compared with the effort required in modal and influence coefficient balancing procedures. A brief discussion of the Unified Balancing Approach and its relationship to the modal and influence coefficient methods is presented. A series of tests which were performed to evaluate the effectiveness of various balancing techniques is described. The results of the Unified Balancing Approach tests are presented and discussed. These results confirm the superiority of this balancing procedure for the supercritical shaft test rig in particular and for multiple-mode balancing in general.

# APPENDIX B

Selected significant references are individually discussed in this appendix, arranged in the same order as they appear in the reference list.

[2] Den Hartog, J. P., Mechanical Vibrations, McGraw-Hill Book Company, Inc., New York, 1934.

The material in these articles is somewhat dated. However, a good overview of the art of rotor balancing is presented, as it existed in the 1930s. The first two of these articles is devoted to rigid rotor balancing. Descriptions are presented of several methods and machines that were used for rigid rotor balancing at the time of publication. This was prior to the development of precise instrumentation and hard-bearing balancing machines. It is apparent from the discussions in these articles that the lack of precise instrumentation added substantial complication to the rotor balancing procedures that were being used.

The second of these articles presents a method for the balancing of a rigid rotor in two planes, simultaneously. This method is distantly related to influence coefficient balancing, but is extremely restricted in its applicability. It would be appropriate for use with a soft-bearing balancing machine.

The need for balancing of flexible rotors is described in the third of these articles along with an outline for a method for performing flexible rotor balancing, a single plane at a time. A case study is also discussed involving a continually changing unbalance condition in a steam turbine due to uneven heat or cooling, and subsequent thermal distortion, of the rotor.

As this reference is concerned primarily with rigid rotor balancing, it is included for reference only and does not directly relate to the Unified Balancing Approach.

[8] Rieger, N. F., and Crofoot, J. F., Vibrations of Rotating Machinery, Part I: Rotor-Bearing Dynamics, The Vibration Institute, Clarendon Hills, Illinois, 1977.

In this chapter from their book on rotordynamics, the authors began by discussing the nature of unbalance and its effect on the synchronous vibration of rotating machinery. A list of typical sources of rotor unbalance is presented along with a classification of rotors (according to ISO document 1940). A thorough discussion of rigid rotor balancing is presented, including the theory behind, the criteria for and some machines used for rigid rotor balancing. This discussion is relatively up to date (1977), particularly with respect to the development of automated balancing machines.

A thorough discussion of flexible rotor balancing is also presented. A basic description of and classification system for flexible rotors are presented. This is followed by a discussion of modal balancing theory and procedures, including a brief history. A discussion of influence coefficient balancing is also presented. The theory is developed for both the exact-point and least-squares procedures. At one point in the development the authors stated that the influence coefficients are real, while at another point that they are complex. This is an apparent contradiction. An historical review of the literature on flexible rotor balancing is presented. Flexible rotor balancing criteria are reproduced from ISO document N18. These balancing criteria are also applicable to rigid rotors. This chapter is concluded with a fairly extensive list of references.

This reference includes discussions of current flexible rotor balancing methods whose disadvantages relative to the Unified Balancing Approach have been discussed in detail.

[12] Rieger, N. F., "Flexible Rotor-Bearing System Dynamics: III. Unbalance Response and Balancing of Flexible Rotors in Bearings," ASME Monograph, 1973.

In this monograph, the author presented a detailed review of the literature on unbalance response and balancing of flexible rotors, as it existed in 1973, along with analytical outlines of the "state-of-the-art" approaches. The paper is effectively separated into two distinct parts, the first dealing with unbalance response and the second with flexible rotor balancing. As the latter of these subjects is the central topic in this study, the balance of this review is devoted to the corresponding portion of this paper.

The author began with a brief discussion of the motivation for flexible rotor balancing. He began a discussion of the literature by referring to some of the more practically oriented papers, including some of the earliest ones. A brief analytical background of modal balancing is presented, with frequent references to the literature, particularly that of Bishop and his coworkers at University College in London. The analytical basis for influence coefficient balancing is also presented, again with frequent reference to the literature. Mention is also made of several other analytical balancing procedures which are presented in the literature, although none of the analysis is included. Some special mention is also made of some of the foreign language contributions to flexible rotor balancing, in particular those from Czechoslovakia and Russia.

The author then proceeded to make an assessment of the state of the art of flexible rotor balancing. He stated that, in practice,

most flexible rotor balancing is accomplished by one of several trial-and-error methods and that no systematic procedure has gained wide acceptance. He also noted the lack of any true flexible rotor balancing machine and standard (as of 1973) as being a hindrance to the practical acceptance of a systematic procedure. The author concluded that the development of some type of a computerized flexible rotor balancing system is inevitable.

The author stated that although modal balancing is the most developed of the flexible rotor balancing procedures, it is fraught with certain disadvantages, which are enumerated in the paper. First, the requirement for predicted mode shapes and natural frequencies is not only an inconvenience with this procedure, but can be a source for considerable error. Second, some of the modal techniques are rendered ineffective if vibration data cannot be acquired in close proximity to the resonant peaks. Third, no general purpose modal balancing computer program is in existence (as of 1973) and, more importantly, none of the modal procedures appears to lend itself to programming. Thus, the development of a practical user-oriented flexible rotor balancing system utilizing a strictly modal approach does not appear to be feasible.

The author then evaluated the influence coefficient method, with considerably more optimism. He stated that, although the number of publications on influence coefficient balancing is considerably less than that for the other methods, this procedure has certain inherent advantages, particularly the convenience of application and straightforward adaptability for an automated balancing system. In the

author's opinion the only factors which limit the influence coefficient method are the precision of the vibration data, the occurrence of non-circular rotor whirl orbits and variations in the test setup (e.g., changes in speed during data acquisition). The author considered that these limitations could be handled with analytical and statistical procedures. The author was apparently unaware (as of 1973) of some of the other problems which can occur in influence coefficient balancing, such as when redundant balance planes result in the calculation of impractical correction masses and the frequent difficulty involved in obtaining useful trial mass data at speeds above one or more critical speeds of a lightly damped system.

According to the author, the "other methods" referred to in the literature are generally variations of modal balancing, and are designed for and useful for specific applications only.

The author concluded by restating briefly the state of the art in flexible rotor balancing and the need for the development of an automated flexible rotor balancing system. Although paying lip service to modal balancing, the author is clearly a proponent of influence coefficient balancing. This monograph presented a fine review of the literature which was, and is, sorely needed in the field of flexible rotor balancing.

The advantages of the Unified Balancing Approach with respect to the methods described in this reference (influence coefficient and modal balancing) have been discussed in considerable detail.

[16] Kroon, R. P., "Balancing of Rotating Apparatus--I," ASME Transactions, Journal of Applied Mechanics, Vol. 10, p. A-225, 1943.

This is the first of a two-part series on rotor balancing, intended primarily as a design guide. The author began this part by defining rotor balancing, unbalance, and some of the reasons for the practical necessity of rotor balancing. He made the important observation that the vibratory system of interest includes not only the rotating hardware, but often much of the stationary hardware, as well.

This introduction is followed by a detailed theoretical analysis of a narrow, rigid disk mounted symmetrically on a flexible, massless shaft, which is in turn supported on rigid bearings. This model is sometimes referred to in the literature as a Jeffcott rotor. A typical response curve is presented along with a definition of critical speed. The extension of this model to include flexible bearings is considered, including the possibility of anisotropic supports, for which separate critical speeds would be observed in orthogonal directions (e.g., vertical and horizontal).

The theoretical basis for rigid rotor balancing is presented, with individual attention being given to the specific cases of "static" and "dynamic" balancing. A discussion of flexible rotors follows, with emphasis on the inadequacy of rigid rotor balancing for flexible rotors. The author concluded with an illustration in which he used Rayleigh's reciprocal theorem to infer the size of a correction mass required to compensate for a known unbalance for a particular flexible shaft. This is an interesting exercise, but is apparently of practical

value only if vibration measurements can be made precisely at the balancing plane locations. If this condition can be met, a substantial reduction in the number of individual trial mass runs may be possible using the Unified Balancing Approach.

In general, this paper is a good introduction into synchronous rotor vibration and rotor balancing and provides a suitable background for the second paper in this series, which is a discussion of specific rotor balancing machines and methods, as they existed in 1944.

While briefly mentioning flexible rotor balancing, no methods were discussed which could be compared to the Unified Balancing Approach.

[17] Kroon, R. P., "Balancing of Rotating Apparatus--II," ASME Transactions, Journal of Applied Mechanics, Vol. 11, pp. A-47 - A-50, 1944.

This is the second of a two-part series on rotor balancing, intended primarily as a design guide. The first part was a review of the basic principles of synchronous whirl and rotor balancing. This part is a discussion of specific rotor balancing machines and methods, as they existed in 1944. The author began with a description of shop balance methods. Soft-bearing balancing machines are discussed, with particular emphasis on movable-fulcrum machines. The author described the method of correction mass determination for the case when the fulcrum positions coincide with the balancing plane locations and the somewhat more complex correction mass calculations that are required when the fulcrum positions do not coincide with the balancing plane locations.

A method is described for field balancing that is precisely a two-plane, one-speed, exact-point influence coefficient procedure. As the author mentioned, this method may, of course, be used for shop balancing as well, and does not require any fulcrums or locking bearings. Both graphical and analytical solution methods are described. A numerical example is given to illustrate the calculations procedure. The author stated that this analytical solution method can be applied to rotor systems with more than two bearings (and consequently more than two balancing planes), but that it is generally "more convenient to find the corrective weights by trial and error rather than by solving complicated determinants." Presumably, by "trial and error"

he was referring to graphical manipulations rather than physical manipulation of correction masses. In any case, the advent of the high-speed digital computer has substantially altered this situation.

The author concluded with some general observations about practical field balancing problems which are very important and are usually overlooked. He stated that while balancing procedures assume linear rotor behavior, this is not always the case, and that under certain circumstances more than a single balancing process may be required, possibly including a redetermination of influence coefficients. The importance of taking all balancing data under consistent operating conditions is stressed. The author considered some situations in which it may be advisable to either change the balancing plane locations or to increase the number of balance planes. Of particular interest is the case when one or more of the balancing planes is located at (or very near to) a vibration node and the case when the influence coefficient matrix is (nearly) singular. The author warns that in order to control local resonances (such as pedestal resonances) it may be necessary to detune the part in question. In cases where the synchronous vibration is variable (with time) certain compromises may be necessary, according to the author. This is a very informative and clearly written paper, particularly with regard to the discussion of practical field balancing problems.

As this reference is concerned primarily with rigid rotor balancing, it is included for reference only (particularly with regard to field balancing) and does not directly relate to the Unified Balancing Approach.

[18] Muster, D., and Flores, B., "Balancing Criteria and Their Relationship to Current American Practice," ASME Vibrations Conference Paper 69-VIBR-60, Philadelphia, 1969.

In this paper, the authors have compiled rigid rotor balancing criteria from a variety of sources. The results of a survey of American industry are also presented, which reflect the balancing conditions in current (1969) practice. Such information as rotor size and weight, balancing speed and unbalance tolerances was solicited. A respectable response to this survey was generated, relating to an annual production rate of almost ten million rotors per year. A series of statistical analyses was performed in an attempt to establish data trends. Though much of the data was biased and not normally distributed (due to insufficient quantities of data), some general trends were observed.

In general, it was found that the balancing criteria used in practice met, or exceeded, the general balancing specifications referenced by the authors (particularly the military specifications). Also, it was noted that in the overwhelming majority of practical balancing situations, the balancing speed was considerably lower (often by more than an order of magnitude) than the running speed of the rotor. It is not unlikely that in a number of these cases, the dynamics of the rotor at the running speed is dissimilar from that at the balancing speed (particularly in terms of shaft bending). In such cases, rigid rotor balancing may not be adequate to produce smooth operation at the running speed, and this effect may not be accurately reflected by the reported unbalance tolerances.

The authors listed forty subclasses for classification of rotors. Although insufficient data were presented from which to draw inferences for the individual subclasses, the authors claimed that as more data are received such inferences can and would be drawn.

As this reference is concerned primarily with rigid rotor balancing, it is included for reference only (particularly with regard to balancing criteria) and does not directly relate to the Unified Balancing Approach.

[19] Jackson, Charlie, "Using the Orbit [Lissajous] to Balance Rotating Equipment," ASME Paper No. 70-Pet-30, (September) 1970.

A very simple method is described for balancing rotating machinery with a single balancing plane, using an oscilloscope lissajous pattern (or orbit). A detailed description is given of the instrumentation required for this procedure and its method of use. This instrumentation includes two inductive displacement sensors "mounted at each bearing and located 90 degrees apart and at the same radius." Another sensor probe is located such that an existing, or artificially created, shaft discontinuity produces a very distinct once-per-revolution signal. A sensor used in this way is often referred to as a keyphasor. The two probes located at one of the bearings are connected to a dual-channel oscilloscope, with one probe connected to the horizontal amplifier and the other probe to the vertical amplifier. The keyphasor output is connected to the trigger input of the scope and the z-axis input of the Cathode Ray Tube (CRT). The oscilloscope then displays an orbit pattern, which "represents the actual motion of the shaft with the bearing." Some typical orbit patterns are illustrated and discussed. The effect on the orbit pattern of the phase relationship between the two sensor signals is also discussed. The keyphasor signal causes a blanking of the orbit which permits the measurement of phase angle relative to the shaft discontinuity. A method is described for determining the phase angle of an orbit.

The balancing procedure described involves the identification of the angle of the rotor unbalance (in a single plane only) as

coinciding with the phase angle of the orbit. This, of course, requires that the phase angle measurement be made while the rotor is running at a speed substantially below its first critical speed. The advantage of this balancing method is that no trial mass run is required. However, since no indication is given of the proper size of the correction mass, the first correction mass behaves essentially like a trial mass. In addition, it is often difficult to get sensitive vibration data at speeds substantially below the first critical speed. Clearly, the single-plane limitation of this method precludes its use for virtually any rotor that runs above one or more rigid or flexural critical speeds. However, this method may be useful for balancing subcritical machinery, in cases where precision instrumentation is not available.

As this balancing method is not suitable for supercritical rotors, it cannot be considered a true flexible rotor balancing method and thus does not directly relate to the Unified Balancing Approach.

[20] Stadelbauer, D. G., "Balancing Machines Reviewed," Schenk Trebel Corporation, n.d.

This paper claims to present a review of the history of balancing machines from their inception at the beginning of the twentieth century to the present (c. 1978). What is in fact presented is a history of rigid rotor balancing machines, as distinguished from the more recently developed general (rigid and flexible rotor) balancing systems.

A brief review of these rigid rotor balancing machines is presented, beginning with the earliest trial-and-error machines and continuing with the more recent compensation, soft-bearing and hard-bearing machines. The relative merits of the soft- and hard-bearing machines are presented with a clear endorsement of hard-bearing machines. A brief mention is made of the introduction of digital computerized electronics to replace and enhance the analog electronics currently in use in hard-bearing machines. No mention is made whatsoever of flexible rotor balancing or flexible rotor balancing systems.

As this reference is concerned exclusively with rigid rotor balancing, it is included for reference only (particularly with regard to balancing machines) and does not relate directly to the Unified Balancing Approach.

[21] Grobel, L. P., "Balancing Turbine-Generator Rotors," General Electric Review, Vol. 56, No. 4, p. 22, 1953.

This paper apparently represents the first published application of modal balancing for flexible rotors (in particular, turbine-generator rotors). The author began by stressing the necessity of having a well-balanced rotor. He discussed the cause and effect of "static" and "dynamic" unbalance, as well as that for "higher forms of vibration" (i.e., flexible rotor vibrations). He went on to explain why rigid rotor balancing is inadequate for many turbine-generator rotors (i.e., they behave flexibly). He also discussed some interesting phenomena (particularly thermal effects) which tend to dynamically alter the rotor unbalance.

The author then proposed a balancing method which he referred to as "precision balancing." While the author did not explain, in detail, how this balancing method is used, it appears that it is, in large part, trial and error. The object of the procedure is to balance the rotor a single mode at a time, using sets of correction masses (i.e., modal balancing). However, it is not clear precisely how this is to be accomplished. The author presented some data to indicate the practical advantages (in terms of reduced on-site balancing effort) gained by the introduction of "precision balancing," as compared to the exclusive use of rigid rotor balancing.

While the application of this balancing method may have been somewhat crude, there is no question of its significance in that it introduced the concept of modal balancing. The advantages of the

Unified Balancing Approach over modal balancing methods in general (and, thus, Grobel's method in particular) have been discussed in considerable detail.

[25] Bishop, R. E. D., and Gladwell, G. M. L., "The Vibration and Balancing of an Unbalanced Flexible Rotor," Journal of Mechanical Engineering Science, Vol. 1, No. 1, p. 66, 1959.

This is the first paper in which modal balancing (as defined by Bishop and coworkers) was proposed. The authors began by deriving the equations of motion for a uniform rotor in radially rigid (and angularly free) bearings with both external and internal damping. They went on to show that the solutions to these equations are in the form of a series of the characteristic functions of the shaft. These characteristic functions are shown to be orthogonal to one another and to represent the modal responses for the various undamped resonances of the shaft. Using a similar series to represent the mass eccentricity of the shaft, the modal components of the rotor response are shown to be directly related to the corresponding modal components of the eccentricity (and orthogonal to the noncorresponding components of eccentricity).

The authors then analyzed the effect of conventional "low speed" balancing. While the object of such balancing is to eliminate the forces and moments imposed on the bearings at the balancing speed, the effect at other speeds, for a flexible rotor, is quite unrelated and can be detrimental, as illustrated by an example presented in the paper.

The authors then described a procedure (in very general terms) for balancing flexible rotors which takes advantage of the orthogonality of the characteristic functions to permit balancing of a shaft mode by mode. It is presumed that the modal response of the shaft (at

least at a selected number of axial locations) can be determined empirically for the mode being balanced, as well as for all previously balanced modes, although it is not clear how this is to be done. Then, assuming planar mode shapes, a planar set of correction masses can be found which will balance the mode of interest while not affecting the previously balanced modes. The method proposed by the authors for determining the magnitude, and angular orientation of the correction mass set involves a combination of operator intuition and trial and error.

The authors conclude the paper with analytical discussions of the effects of shaft bow and static gravity sag on the unbalance response of a uniform rotor. Both effects were found to be, generally, small.

Clearly, this is a very important paper as it introduces modal balancing and develops the modal balancing objectives, which have been adapted for the UBA. There are a number of disadvantages to this balancing method (and to modal balancing in general), which have been discussed at length in Chapter 4. Briefly, this method assumes planar mode shapes and light damping (which are not always the case), and requires substantial operator insight (particularly as outlined in this paper). Also, it presumes (as described in this paper) that the rotor vibration at a particular speed can be completely characterized by measurement at a single axial location. Part of the justification for the UBA is to remedy these shortcomings while maintaining the basic objectives of modal balancing.

[35] Bishop, R. E. D., and Parkinson, A. G., "On the Use of Balancing Machines for Flexible Rotors," Journal of Engineering for Industry, Vol. 94, p. 561, 1972.

A discussion of flexible rotor balancing methods is presented, with particular emphasis on the question of whether the rotors should be balanced as rigid bodies prior to the application of the flexible rotor balancing procedures. The authors began with a brief recitation of the motivations for and methods of flexible rotor balancing. The discussion is essentially restricted to modal balancing methods, as the (unnamed) influence coefficient method is disposed of with the comment that it is an "indirect" approach while "we are concerned with a systematic direct approach to the problem [and] we have no doubt that an adequate, direct solution is far more desirable." It is not entirely clear in what respect the authors considered the influence coefficient method to be indirect, and this method is, if anything systematic.

The authors continued with a brief review of modal balancing theory, liberally sprinkled with useful references. In a discussion on the object of balancing, mention is made of the occasional need to control rotor amplitudes, but the primary concern is with the reduction of bearing forces. The authors claimed that, for any particular mode, the modal distribution of unbalance can be easily obtained, but did not state how this is to be done. Presumably, this can be found in one of the references, although no specific citation is given.

The general class of "modal" balancing procedures is divided into two groups. The first group, which includes the method advocated

in this paper, requires the use of N balancing planes for balancing the $N^{th}$ mode (without rigid rotor balancing) so as not to disturb the (N-1) lower modes. The second group requires the use of (N+2) balancing planes for balancing the $N^{th}$ mode (after rigid rotor balancing), so as not to disturb either the rigid rotor balance condition or the (N-1) lower modes. In general, the authors concluded that, while the (N+2) plane method may sometimes result in lower bearing forces than the N plane method (though generally not a significant difference), the N plane method has no trouble meeting practical balancing criteria and the additional cost of the (N+2) plane method (in equipment, time and labor) is not justified. An averaging technique, using two additional balancing planes, is discussed which the authors claimed will bring the results of the N plane method at least to parity with the (N+2) plane method (if not better) even in extreme situations.

Discussions were contributed by Dr. W. Kellenberger and Prof. K. Federn in which attempts were made to refute the authors' contention that the additional cost of the (N+2) plane method is not justified. The paper is concluded with a rebuttal by the authors in which they restated their case.

The advantages of the Unified Balancing Approach over modal balancing methods in general (and the N plane method in particular) have been discussed in considerable detail.

[39] Dimentberg, F. M., "Theory of Balancing Flexible Rotors," Russian Engineering Journal, Issue No. 11, 1964.

The author began with a brief discussion of the need for and purposes of flexible rotor balancing. He stated that in most cases, the objective of flexible rotor balancing, as for rigid rotor balancing, is to reduce or eliminate the bearing forces. He does mention the occasional need to consider the effect of "bending forces and rotor strength or need to maintain the clearances between the rotor and other elements." However, he apparently considered the use of this type of balancing criteria as principally supportive to the use of bearing forces.

In the remainder of this article, Dimentberg reviewed, what he considered to be, the most important recent developments in the field of flexible rotor balancing. The literature reviewed is mostly Russian and is, without exception, concerned with modal balancing in one form or another. Dimentberg concluded by mentioning areas that require further study and by observing that the development of experimental methods is essential for the application of flexible rotor balancing techniques. This last point is a very important one that is often overlooked in the literature.

The modal balancing methods discussed by Dimentberg have the same disadvantages as the modal balancing methods described in Chapter 4. The advantages of the Unified Balancing Approach relative to such modal methods have been discussed in considerable detail.

[42] Kellenberger, W., "Balancing Flexible Rotors on Two Generally Flexible Bearings," Brown Boveri Review, Vol. 54, No. 9, pp. 603-617, 1967.

In this paper, the author introduced the (N+2) plane modal balancing method, which he referred to as the "method of minimum planes." He derived the normal mode representation for the synchronous whirl of a general, undamped rotor, including the effects of discrete and continuous unbalance. He also stated that this representation is valid for rotors supported by two linear, generally flexible, bearings (i.e., including the effects of stiffness, damping, and mass). Based on this modal representation, a method is proposed ("method of minimum planes") for rigid (low speed) and flexible rotor balancing. Numerical examples are presented to demonstrate the procedures and accuracy for this balancing method. In an appendix, the author proved analytically the orthogonality of the normal modes.

Like Bishop and Gladwell [25], Kellenberger presented a series of equations representing the normal mode unbalance which must be satisfied to achieve a balanced rotor. However, unlike Bishop and Gladwell, he insisted that it is also necessary to satisfy two additional equations which represent the rigid body unbalance of the rotor; thus, the necessity for at least (N+2) planes for balancing through the $N^{th}$ mode. The use of this balancing method requires prior knowledge of the mode shapes (eigenfunctions) of the rotor. The author's procedure used analytical predictions to provide this information. He also stated that, while the relationships between the (N+2) balancing masses can be exactly determined, the actual size and

orientation of this mass set must be determined using trial-and-error tests. However, he did not specify the form of these tests.

While this method looks good in theory, a number of practical questions have not been addressed. These include the effect of inaccuracies in the analytical predictions on practical balancing results; the effect of residual unbalance in the balanced modes; the means by which data are acquired (including practical linearity and resolution limitations); and the form of the trial-and-error tests. While the numerical examples presented by the author are impressive, they are of only limited value, as real rotors tend not to behave "perfectly" and experimental results are needed to truly verify any rotor balancing method.

Some of the questions discussed above, as well as other disadvantages of this balancing method (inherent in modal balancing) are eliminated by the Unified Balancing Approach.

[43] Kellenberger, W., "Should a Flexible Rotor Be Balanced in N or (N+2) Planes?" Journal of Engineering for Industry, Vol. 94, p. 548, 1972.

The author presented a case to support his claim that the (N+2) plane method for balancing flexible rotors is more effective than the N plane method. The (N+2) plane method involved prebalancing the rotor as a rigid body, followed by the use of (N+2) balancing planes for balancing N modes. By contrast, the N plane method did not involve any rigid rotor balancing and requires the use of only N balancing planes for balancing N modes. Both methods were clearly based on modal balancing principles.

The author's thesis was based on the contention that the object of flexible rotor balancing is the elimination of rotating bearing forces and essentially ignored the consideration of rotor amplitudes. He stated the conditions for the elimination of rotor deformation (amplitudes), but then goes on to the conditions for eliminating bearing forces with the apparent (unstated) conclusion that N planes are perfectly sufficient to eliminate the deformation, while the two additional planes are necessary only for the elimination of the bearing forces.

A "practical procedure" was described and illustrated for two theoretical cases. This procedure, as described, required the use of analytically determined mode shapes, which would appear to detract from the practical applicability of the method. The author presented two numerical examples, involving an "ideal" uniform rotor on rigid supports with simple, planar unbalance distributions, to support his

claim of the superiority of the (N+2) plane method. The conclusion was drawn that the increased effectiveness of the (N+2) plane method (in terms of reduced bearing forces) justified the additional effort.

This paper was appended with a discussion by Professor R. E. D. Bishop and Dr. A. G. Parkinson, and a rebuttal by the author. The discussers attempted to refute the conclusions drawn by the author, particularly with respect to the superiority of the (N+2) plane method. They disputed the value of "simplistic" numerical examples, but went on to present one of their own. Their principal claim was that their modal balancing method (which is essentially an N plane method) is superior to the method used in the author's examples and would fare better when compared to the (N+2) plane method. Any subsequent difference between their N plane method and the (N+2) plane method would not justify the additional cost (and effort) of the (N+2) plane method.

In his rebuttal the author began by citing a calculation error in the numerical example presented by the discussers and claimed that the correct results from this example support his position. He continued by restating his conclusions, including the point that, although some situations may exist for which the additional effort of the (N+2) plane method is not justified, in general this is not the case.

The numerous advantages of the Unified Balancing Approach over modal balancing in general (and Kellengerger's method in particular) have been discussed in considerable detail.

[46] Lund, J. W., "A Method for Using the Free Shaft Modes in Rotor Balancing," Vibrations in Rotating Machinery, Institute of Mechanical Engineers' Conference Publications 1976-9, pp. 65-71.

The author introduced an interesting new approach to flexible rotor balancing which is a variation of modal balancing. The principal advantages of this procedure over more conventional modal methods are that the usual restrictions to lightly damped rotors with planar modeshapes do not apply.

The author began with the reasonable assumption that "the anisotropy and the damping of the system are confined to the bearings and the supports, while the shaft by itself is a conservative system with orthogonal mode shapes." An accurate analytical model of the free shaft is an essential requirement of this balancing method. The author claimed thatsuch a model is initially derived from the known dimensions and material properties of the shaft and then adjusted so as to make it yield the same natural frequencies as those observed in tests with the rotor softly suspended. However, duplicating measured natural frequencies does not guarantee that the model is an accurate representation of the rotor. In addition, it may not always be possible to perform the tests recommended by the author to measure these natural frequencies.

The author went on to describe, in detail, the procedures and analyses required to balance rotors with either isotropic or anisotropic bearings. He stated that the method for the isotropic case is also applicable to the anisotropic case if the effect of the gryoscopic moments can be considered to be negligible. In either

case, the number of trial mass runs is, at most, equal to the number of bearings. If the rotor has been previously balanced as a rigid rotor, as few as zero trial mass runs may be required.

The author demonstrated this balancing method with a numerical example. This example includes the effect of measurement error, but not of modelling error (the model is exact). The results of this example are impressive, but as the author stated, "In the end, only experiments can decide whether the proposed method of balancing works in practice, and for this purpose a test program is planned for the near future." The results of this test program should prove to be very interesting.

In general, the proposed balancing method seeks to expand the range of applicability of modal balancing and accelerate the procedure by reducing the required number of trial mass runs. However, the practical applicability of this method may be adversely affected by the requirement for "a sufficiently accurate analytical model for the shaft." Not only is it often difficult to obtain such a model, but many balancing engineers consider the requirement of an accurate rotor model as an unattractive feature of a balancing method. Since any balancing method is of limited value if it is not accepted, the most important practical consideration is operator acceptance. The degree of sensitivity of this balancing method to rotor model inaccuracies has yet to be determined.

By comparison, the Unified Balancing Approach has the advantage of not requiring analytical mode shape predictions.

[47] Giers, A., "Practice of Flexible Rotor Balancing," Vibration in Rotating Machinery, Institute of Mechanical Engineers' Conference Publications 1976-9, pp. 33-42.

The author presented a discussion of several methods for balancing flexible rotors from the viewpoint of practical application. These methods include the N and (N+2) modal balancing procedures (referred to here as "modal balancing" and "comprehensive modal balancing," respectively) as well as influence coefficient balancing.

The author first constructed a criterion for comparison of balancing methods using the ratio of the sum of the bearing forces (due to residual unbalance) to the weight of the rotor. He then evaluated each of the modal balancing methods using a numerical example. However, he never really made any direct comparison between the balancing methods (including influence coefficient balancing) or stated clearly for what conditions each would be most appropriate. Also his treatment of the modal balancing methods drew somewhat heated discussions from Bishop and Parkinson.

The author's discussion of influence coefficient balancing is primarily a reiteration of the conclusions of Tessarzik [65]. The author added a comment to the effect that Tessarzik's results could have been improved thorugh the use of modal balancing, a conclusion which was strongly disputed in a discussion by Tessarzik. The author also stated that an improved method could be devised from a combination of modal and influence coefficient balancing, but did not indicate how this would be accomplished. He did mention that it is

advisable not to put "blind trust in the computer," which is an indication of the numerical difficulties to which influence coefficient balancing is susceptible.

The paper continues with the description of an automated data acquisition system for a rigid rotor balancing machine. While practical data acquisition is of considerable importance in the balancing of flexible rotors, the relevance of this particular example to the central topic of the paper is not clear. Also, while the purpose of this paper is presumably to provide the practitioner with practical guidelines for selection, use and improvement of the existing flexible rotor balancing techniques, the result of the paper appears to have been primarily to resurrect an old debate.

The advantages of the Unified Balancing Approach over all the methods considered in this reference have been discussed in considerable detail.

[48] Thearle, E. L., "Dynamic Balancing of Rotating Machinery in the Field," ASME Transactions, Journal of Applied Mechanics, Vol. 56, pp. 745-753, 1934.

The author described a method for in situ balancing of rotating machinery in two planes, simultaneously. This method is essentially a form of influence coefficient balancing that is restricted to two planes. In fact, this paper may represent the earliest efforts at influence coefficient balancing (1934). The procedure described employs a combination of analytical and graphical techniques. This procedure is outlined in step-by-step detail and a procedure sheet is presented as an aid. A simplified procedure for trim balancing (i.e., using predetermined influence coefficients) is also described.

A "balancing machine" for use in field balancing is described. This "machine" is simply an instrument for providing the vibration data used for the calculation procedure presented in the first part of this paper. This instrument is composed of a pair of linear generators, one of which is attached to each bearing pedestal of a two-bearing machine in such a way that the voltage output of the generators is proportional to the bearing pedestal vibration. It is noted in the paper that balancing of a three-bearing machine requires an iterative procedure using the vibration data from two bearing pedestals at a time. With general influence coefficient balancing (as later developed), such an iterative procedure is, of course, not required.

The Unified Balancing Approach possesses neither the limitations of Thearle's method nor the practical disadvantages of the subsequently developed general influence coefficient procedures.

[49] Baker, J. G., "Methods of Rotor-Unbalance Determination," ASME Transactions, Journal of Applied Mechanics, pp. A-1 - A-6, (March) 1939.

This paper is presented in three parts. In the first part, the author describes a "conventional" balancing method, which is, in essence, influence coefficient balancing where the number of balance planes must equal the number of vibration reaadings. He discusses the errors inherent in the balance mass calculations under certain conditions. These conditions can occur as a result of using non-independent balance planes. He then proposed a modified technique which utilizes trial mass sets which are designed to effect only a single vibration sensor reading. However, the calculation of the trial mass sets has the same inherent sources of error as the "conventional" procedure. Also, this method is incompatible with modal theory and depends upon distortion of the mode shapes.

In the second part of this paper, the author describes a method for applying "conventional" balancing where the balancing corrections are placed in completely selected locations. This approach is illustrated in terms of both a mathematical algorithm and a balancing machine.

In the third part of this paper, the author discusses center of percussion balancing, which involves mounting a rotor in a rigid, flexibly supported frame with two vibration pickups. These sensors are located in such a way that each of the sensors reacts to unbalances from only one of the correction mass planes. This technique is applicable only for rigid rotor balancing.

A discussion of this paper was contributed by T. C. Rathbone. The writer describes a graphical method of performing the group trial mass calculations proposed in this paper.

The balancing methods described in the reference are all either unsuited for or severely limited with flexible rotors. These limitations do not exist for the Unified Balancing Approach.

[51] Goodman, T. P., "A Least-Squares Method for Computing Balance Corrections," ASME Transactions, Journal of Engineering for Industry, Vol. 86, No. 3, pp. 273-279, (August) 1964.

The author described in detail the empirical and analytical procedures for performing least-squares influence coefficient balancing. He began by citing the motivation for the development of the least-squares method. This is followed by a brief description of the conventional procedure for field balancing (i.e., exact-point influence coefficient balancing).

The mathematical background for the plain least-squares and weighted least-squares procedures is presented. The author stated that, in general, no more than three iterations are needed with the weighted least-squares method to obtain sufficient convergence. The purpose of the weighted least-squares method is to reduce the maximum predicted residual vibrations at the expense of the root mean square (rms) of these residuals. The first iteration of this procedure is identical to the plain least-squares method. The author provided some guidelines for selecting the optimum correction mass set from those calculated using the weighted least-squares method. Some suggestions were also made concerning the selection of balancing planes. In particular, it was recommended that the number of balancing planes be equal to "the number of critical speeds below the maximum balancing speed" (as cited from a reference). The author also stated that the most effective locations for the balancing planes may be determined either experimentally or analytically.

The required input data for and the resulting output data from the analysis for these balancing methods are described. Particular mention is made of the relationship between the sign conventions of the vibration phase angle measurements and the angular locations of the correction masses. A sample problem is presented to illustrate the weighted least-squares procedure. In an appendix the author presented, in detail, the calculations required to perform these balancing procedures.

In a valuable discussion, contributed by Frank J. Heymann, a similar least-squares balancing procedure, which includes some additional features, is described. In lieu of the author's weighting procedures, this method includes a procedure to minimize the sum of the variances of the residual vibrations based on assumed variances of the influence coefficients. The discusser claimed that this procedure tends to "inhibit solutions with large mass corrections in general" and particularly in the case of nearly dependent balancing planes. However, he did not state how these assumed variances are arrived at. This method provides for the identification of the balancing plane which contributes the "least to the net reduction in the quantity to be minimized" and the subsequent calculation of "a second solution in which that plane is omitted." This procedure can be repeated until a single plane remains. In addition, this method provides for the calculation of specific correction mass "plugs" based on "the number of holes, their angular positions, the maximum weights of balance plugs, and the locations and weights of plugs currently installed."

The discusser concluded with some specific observations concerning the application of a balancing procedure of this type. He stated that a systematic approach is necessary, which requires some reorientation on the part of balancing engineers. Also, the input to the procedure should be "simple and quick" so as not to discourage its use and "ready access to the computer and quick turn-around time is essential" (a problem which can be eliminated by the use of mini and microcomputers). The discusser suggested that the most desirable approach toward establishing computerized production balancing be a streamlined procedure designed to meet the specific needs of the particular balancing problem while avoiding excessive complication and flexibility.

The author closed by adding that his balancing method, as it was implemented, permits the use of specific selected vibration readings for the calculation of correction masses while still calculating predicted residual vibration levels for all of the vibration readings.

The advantages of the Unified Balancing Approach over influence coefficient balancing have been discussed in considerable detail.

[55] Lund, J. W., and Tonneson, J., "Analysis and Experiments in Multiplane Balancing of Flexible Rotors," ASME Transactions, Journal of Engineering for Industry, Vol. 94, No. 1, p. 233, (February) 1972.

The multiplane-multispeed influence coefficient balancing procedure is described in detail along with the results of a series of tests designed to evaluate this technique. This balancing procedure handles large quantities of vibration data through the use of a least-squares minimization algorithm. The derivation of and equations for this algorithm are presented in one appendix. An algorithm to compensate for data error through the use of additional trial mass runs (for a total of two runs per balance plane) is presented in a second appendix.

An extensive series of tests are described and the results of these tests evaluated. These tests involved the balancing of a flexible rotor through three critical speeds while varying sensor type, instrumentation, balance speeds and balance planes. In general, the variation of the results from test to test was insignificant while the effectiveness of this influence coefficient balancing procedure was clearly and uniformly demonstrated.

The authors concluded that this balancing procedure is valid and accurate and that "for satisfactory balancing of the rotor, it is necessary to measure vibration amplitude and phase angle with an error not exceeding 3 to 4 percent." In other words, the results of the balancing procedure is not likely to be adversely affected to any significant extent by reasonably small instrumentation errors.

This paper presents a very clear detailed description of and guide to influence coefficient balancing. No mention was made in this paper of any of the difficulties since encountered with influence coefficient balancing. These difficulties and the corresponding advantages of the Unified Balancing Approach are discussed in considerable detail in Chapter 4.

[58] LeGrow, J. V., "Multiplane Balancing of Flexible Rotors--A Method of Calculating Correction Weights," ASME Third Vibration Conference Paper No. 71-Vibr-52, Toronto, Canada, (September) 1971.

A procedure for balancing flexible rotors is presented, which is closely related to influence coefficient balancing. The author began with a brief discussion of the purpose of rotor balancing and the complexities introduced by the advent of flexible rotors. He claimed that the weighted least-squares influence coefficient procedure is not practical due to the number of runs (and, consequently, the length of time) required to balance a rotor. In order to reduce this balancing time, the author's method utilizes analytically determined influence coefficients.

Examples are presented of a model rotor and a turbine-generator rotor which have been balanced using the author's method. The critical speeds, mode shapes and influence coefficients of these rotors were calculated using Stodola's Method of Matrix Iteration. Several simplifying assumptions were made during the course of this analysis. These assumptions include neglecting damping, gryoscopic mass moments of inertia, rotatory inertia and cross-coupling bearing effects. Thus, the calculated mode shapes are planar and the calculated influence coefficients are real (as opposed to complex). The equations are presented for the calculation of correction masses for a three-probe, three-plane (exact-point) balancing case using the analytical influence coefficients. Some test results are presented for a model rotor and turbine-generator rotor to verify this balancing

approach. This method was apparently reasonably effective for balancing these two rotors.

The principle advantage of this balancing technique is the reduction in required rotor spin-ups due to the elimination of the trial mass runs. However, in the (not necessarily uncommon) situations when the assumptions listed above are violated or the rotor is too complex to be accurately modeled, the loss of accuracy of this balancing method may not be tolerable. In general, extreme care must be taken when using any analytically derived quantities as an essential component in a rotor balancing procedure. The author claimed that an exact-point procedure (in particular, his exact-point procedure) is preferable to a weighted least-squares procedure because an exact solution is preferable to an iterative solution. He, apparently, has missed the entire point of the weighted least-squares procedure, which is to provide a capability to minimize a large number of vibration readings without requiring an excessive number of balancing planes. These disadvantages of LeGrow's method have been eliminated by the Unified Balancing Approach.

[62] Tessarzik, J. M., Badgley, R. H., and Anderson, W. J., "Flexible Rotor Balancing by the Exact Point-Speed Influence Coefficient Method," ASME Transactions, Journal of Engineering for Industry, Vol. 94, No. 1, pp. 148-158, (February) 1972.

The authors presented the results of a series of tests designed to confirm the validity of the exact point-speed influence coefficient method for the balancing of flexible rotors. The motivation for the development of such a balancing method is presented along with the associated theory. The computational procedure involved in the exact point-speed influence coefficient method is described in detail in an appendix. This method is actually a particular case of the general least-squares influence coefficient procedure, for which the total number of vibration readings (speeds times sensors) must be equal to the number of balancing planes.

The authors describe a test rig with a flexible shaft, supported by two tilting-pad journal bearings, with three heavy disks. One of these disks was located between the bearings while another disk was overhung at each end of the shaft. The test rig was designed to run through two rigid-body and one flexural critical speed. Before assembly the test shaft was balanced on a typical commercial hard-bearing balancing machine. Subsequently, negotiation of the rigid-body critical speeds was not a problem. However, it was not possible to negotiate the flexural critical speed without providing some in-place balancing.

Three balancing tests were conducted, each with a different initial unbalance configuration. In each of the tests, the effectiveness of the exact point-speed influence coefficient method for balancing

flexible rotors was verified. Throughout these tests, emphasis was placed on the practical aspects of applying this balancing technique. The authors attributed a portion of the success of these tests to the accuracy of their instrumentation system. For each of these tests several balancing runs were required, each with individual trial mass runs. The principal advantage of the Unified Balancing Approach over the method reported in this reference is that there is a reduction in the number and an improvement in the sensitivity of the individual trial mass runs. Most of the problems associated with influence coefficient balancing would not have occurred during these tests since only a single flexural mode was balanced.

[64] Tessarzik, J. M., and Badgley, R. H., "Experimental Evaluation of the Exact Point-Speed and Least-Squares Procedures for Flexible Rotor Balancing by the Influence Coefficient Method," ASME Transactions, Journal of Engineering for Industry, Vol. 96, No. 2, pp. 633-643, (May) 1974.

The subject of this paper is an extensive experimental investigation, of which the first stage was reported in an earlier paper [62]. That first series of tests was concerned with the verification of the exact point-speed influence coefficient method for balancing of flexible rotors. A similar series of tests is described in this paper in which the same test rig was used to verify and compare the exact point and least-squares procedures for influence coefficient balancing. In addition, this second set of tests considered other rotor configurations and more difficult balance conditions.

Some theoretical advantages were suggested for each of these balancing procedures. However, the test results indicated that the two methods were, in general, equally effective for balancing this particular test rig and that there was no discernible difference in the results from the two procedures. However, for some specific test cases, there were measurable differences in the results from the two procedures. The authors have attempted to draw some generalized conclusions from these particular results. These conclusions are totally reasonable, and no effort has been made to extrapolate these results beyond their area of relevance.

Two disadvantages of influence coefficient balancing are revealed in this paper. The first of these concern the large, impractical correction masses which may be calculated when the balancing

planes are not all linearly independent (particularly when two balancing planes are located close together). An influence coefficient orthogonalization procedure may be used to detect this condition. Subsequent elimination of the linearly dependent balancing plane(s) will eliminate this problem. The use of the Unified Balancing Approach will also prevent this problem from occurring. For the tests described in this paper, this problem occurred when using the exact point-speed method only and was eliminated when the least-squares method was used. However, there is theoretically no reason why the least-squares method should eliminate this problem in the general case. The second of these disadvantages concerns the large number of trial mass runs that are required for each balancing run (one or two for each balancing plane used). In contrast, when using the Unified Balancing Approach, each balancing run (after the modal trial mass set has been established) requires only one (or two) trial mass runs, in total. For the tests described in this paper, the reduction in effort realized by the use of the Unified Balancing Approach could have, for some of the tests, consisted of as many as ten trial mass runs.

[66] Tessarzik, J. M., Badgley, R. H., and Fleming, D. P., "Experimental Evaluation of Multiplane-Multispeed Rotor Balancing through Multiple Critical Speeds," ASME Transactions, Journal of Engineering for Industry, (August) 1976.

This is the third in a series of papers [62, 64] relating the results of an extended test program designed to test the validity of the influence coefficient balancing method. In this paper, the authors discussed the results of a series of seven test cases (of which at least one was composed of several subexperiments) involving the balancing of a test rig through four bending critical speeds. Of these four critical speeds, two were heavily damped (by the journal bearings) and did not require balancing. The remaining critical speeds (first and fourth) were lightly damped and required balancing to be negotiated.

The seven test cases included variations in several of the test conditions. Several combinations of numbers and types of vibration sensors and balancing planes were used. In general, the effectiveness of the balancing procedure (which was quite good) did not vary. However, two exceptions were noted. First, one attempt was made to balance the fourth critical speed without including any vibration data from the first critical speed (which had been previously balanced). A distinct improvement was observd at the fourth critical speed, but only at the expense of the vibration level at the first critical speed. Although the latter was still negotiable, the detrimental effect was significant and certainly undesirable. In all other cases involving balancing of the fourth critical speed, data form the first critical speed was included and no substantial increase in vibration

level was observed at the first critical speed. Second, an attempt was made to balance the rotor using only vibration data from accelerometers mounted on the bearing housings. Consequently, the data from the accelerometers was reduced substantially, but the rotor vibration was actually increased and negotiation of even the first critical speed was not possible. This was interpreted as being due to a change in the rotor mode shape resulting in nodes being located at or very near to the bearings. The authors mentioned that this is a common danger incurred by using only bearing pedestal vibration data for balancing, which occurs frequently in practice.

In some of the test cases, the initial rotor unbalance configuration was altered. Again, the effectiveness of the balancing procedure did not vary. Balancing tests were run with both rigid and flexible bearing pedestals. The flexible bearing pedestals were designed to include some significant stiffness anisotropy, with the horizontal restraint being somewhat softer than the vertical restraint. This stiffness anisotropy resulted in rather extreme elliptical orbits and two separate first critical speeds (in terms of peak vibration). In all cases, the balancing procedure was very successfully.

In general, these test results showed (apparently for the first time) that the influence coefficient balancing procedure could be used for the balancing of a rotor for operation through more than one bending critical speed. In fact, balancing of a rotor through four bending critical speeds was demonstrated (although only two of the critical speeds were actively balanced). Based on these encouraging

results, the capability of this method for balancing a rotor through any number of flexural critical speeds is inferred by the authors. However, as discussed in Chapter 4, there are a number of difficulties inherent in influence coefficient balancing which may not be apparent when balancing only one or two active modes (such as the tests described in this paper). These disadvantages are eliminated by the Unified Balancing Approach.

[67] Smalley, A. J., Tessarzik, J. M., and Badgley, R. H., "The Stability of an Asymmetric Rotor in Damped Supports," ASME Paper No. 78-GT-172, (April) 1978.

This paper describes the results of a series of analytical and experimental procedures to determine the stability of a rotor with asymmetric shaft flexibility. An analytical method is presented which provides the evaluation of the stability of such a rotor. The results of an experimental investigation, which verify this analytical method, are reported. A brief description is presented of an influence coefficient balancing procedure used successfully during these experimental tests. This balancing procedure is essentially the same as that reported earlier by Tessarzik [66]. This balancing technique is referred to in this paper as the Multiplane Balancing method. These tests involved the balancing of only a single mode and this did not reveal some of the disadvantages of influence coefficient balancing (eliminated by the Unified Balancing Approach) that often occur during multiple mode balancing.

[74] Badgley, R. H., Smalley, A. J., Malanoski, Stanley, Hamm, Robert, and Winn, Leo W., "Rotor Balancing," from Machinery Vibration Seminar Course Notes, Mechanical Technology Incorporated, 1973.

The authors presented a general discussion of rotor balancing with particular emphasis on flexible rotor balancing. Some general motivations for rotor balancing were discussed. Several typical sources of rotor unbalance were listed. A brief discussion of rigid rotor balancing was presented with particular emphasis on limitations. A number of situations were described for which rigid rotor balancing is clearly inadequate.

A graphical example of single plane (influence coefficient) balancing is presented to illustrate the concept. This is followed by a discussion of multiplane balancing, particularly as it relates to flexible rotors. Several of the disadvantages of some of the current flexible rotor balancing techniques were discussed. One particular technique (influence coefficient balancing) was endorsed and the basic theory was explained. The specific theory presented was that of the exact-point method. Additional literature was cited in which this balancing technique was discussed and the results of verification tests were reported.

This was a good introductory article into the field of rotor balancing. The theory that was presented was presented clearly. Some useful practical information is also presented. The material in this article was very basic and was clearly not intended to be very advanced or detailed.

The advantages of the Unified Balancing Approach over influence coefficient balancing have been discussed in considerable detail.

[80] Badgley, R. H., and Tessarzik, J. M., "Balancing of High-Speed Interconnect Shafting for Operation above Multiple Bending Critical Speeds," AHS Paper No. 873, presented at the 30th Annual National Forum of the American Helicopter Society, Washington, D.C., (May) 1974.

A case is made for the use of lightweight power transmission shafting, designed to operate above several bending critical speeds. Specifically, the desires for cost and weight efficiency, as well as the need for simple, reliable designs, are cited as motivations for the development of such supercritical shafting. Some potential applications for supercritical shafting are discussed including specific reference to advanced helicopter and V/STOL aircraft development. Some previous work in the development of supercritical shafting is reviewed. Specific mention is made of a series of tests which demonstrated the general feasibility of supercritical shafting, but failed to produce an adequately balanced shaft using a modal balancing procedure.

A review of the theory of the dynamics of flexible shafting is presented, specifically as related to flexible rotor balancing. A brief description is given of a multiplane-multispeed flexible shaft balancing procedure developed at Mechanical Technology Incorporated.

The results of a computational procedure are presented in which an analytical representation of a supercritical shaft, with a random initial unbalance distribution, was analytically balanced. Separate unbalance response and balancing computer programs were used to generate the balancing correction masses and the subsequent vibration levels of the supercritical shaft. These results showed that even

with very light damping, the balancing of supercritical shafts through multiple critical speeds is feasible. Also, a reduction in the number of balance planes resulted in a reduction of the effectiveness of the subsequent balancing procedure. However, the effect of other factors, such as balance plane location, which may have contributed to this reduction in effectiveness, were not considered.

While this numerical example is quite impressive in favor of influence coefficient balancing, it should be noted that a number of disadvantages of influence coefficient balancing which are important in practical application do not become apparent with numerical examples. These disadvantages do not apply to the Unified Balancing Approach.

[97] Little, R. M., and Pilkey, W. D., "A Linear Programming Approach for Balancing Flexible Rotors," ASME Transactions, Journal of Engineering for Industry, ASME Paper No. 75-DET-94, 1976.

The author described a procedure for balancing flexible rotors that is based on a linear programming approach. This procedure is very similar to influence coefficient balancing except that the number of balancing planes is specified to exceed the number of vibration readings (a condition which is not permitted in conventional influence coefficient balancing).

The author began with a general description of conventional influence coefficient balancing. This is followed by a discussion of the theoretical basis and procedure for his modified approach. The procedure began with the identification of an "objective function" to be minimized, which is generally taken to be the residual vibration at a speed (or condition) at which observations cannot be made (e.g., a speed which cannot be reached due to excessive vibration amplitudes). The minimization of this objective function requires the use of analytically determined (e.g., from a transfer matrix analysis) influence coefficients for that condition. This could possibly be a source of substantial error. The original influence coefficient equation is then used as a constraint equation. An additional constraint may be added to require the sizes of the correction masses to fall within an acceptable range, thus preventing the calculation of excessively large, or small, correction masses. The author stated that even in cases where it is not possible to provide analytical influence coefficients for the objective function, this method still

has the advantage of permitting restrictions on the sizes of the correction masses.

An analytical example problem is presented which illustrates the use of this method and compares it to conventional influence coefficient balancing. The results of the two methods, in terms of the predicted residual vibration levels, are comparable, except for that which is related to the objective function (at a speed where no observation is made). For this case, the author's method provides substantial improvement. However, since an analytical influence coefficient for an analytical problem is exact, this example provides an ideal condition for applying this method. It would be instructive to see this method applied to an actual rotor. Also, the use of large numbers of balancing planes may be precluded by practical limitations. Many of the practical disadvantages of influence coefficient balancing, which are eliminated by the Unified Balancing Approach, still apply to the balancing method proposed in this reference.

[102] Larsson, Lars-Ove, "On the Determination of the Influence Coefficients in Rotor Balancing, Using Linear Regression Analysis," Vibrations in Rotating Machinery, Institute of Mechanical Engineers' Conference Publications 1976-9, pp. 93-97.

A statistical procedure is presented for calculating influence coefficients from multiple sets of trial mass data from the same, or similar, rotors. Statistical inferences are drawn concerning the reliability of the results of this procedure. Thus, for a specific set of correction masses (calculated in the usual fashion) a set of confidence limits can be established corresponding to the predicted residual vibration levels.

The author stated that the conventional assumption in influence coefficient balancing that the trial masses are the only cause of the changes in the rotor responses is not entirely valid in practice since "internal alterations in the rotor . . . give rise to an unknown random variation of the unbalance distributions." He claimed that the effects of this random variation can approach, or even exceed, those due to the trial masses. This can lead to significant errors in the influence coefficient matrix and, subsequently, in the calculated correction masses.

The author presented the theoretical basis for his procedure in some detail. In particular, he discussed the evaluation and interpretation of the coefficient of multiple determination and the coefficient of reliability. The former of these coefficients provides a measure of the correlation between the influence coefficients and the trial mass data, while the latter provides a measure of the reliability of the calculated correction masses.

The author discussed the results of tests concerning two homogeneous groups of actual rotors to which this procedure was applied. One of these groups consisted of 37 small rigid turbo rotors, while the other group consisted of 2 large flexible turbo rotors. These results indicate that this procedure provided an effective means for balancing these rotors and that the resulting influence coefficients can be used for reliable balancing of other rotors in the same class. Thus, when a large group of similar rotors requires balancing, this procedure can provide substantial time and cost savings when balancing the later rotors in the group.

Many of the practical disadvantages of influence coefficient balancing, which are eliminated by the Unified Balancing Approach, still apply to Larsson's method.

[104] Drechsler, J., "A Combination of Modal Balancing and the Influence Coefficient Method," Proceedings of the Fourth World Congress on the Theory of Machines and Mechanisms, Newcastle/Tyne, 1975.

The author proposed a method for balancing flexible rotors which is, in a fashion, a combination of modal and influence coefficient balancing. In essence, this method is an extension of the least-squares influence coefficient procedure where the influence coefficient matrix is modified by the use of a weighting matrix designed to account for the predicted mode shapes of the rotor. Thus, the problem of ill-conditioned influence coefficient matrices (e.g., due to the use of redundant balancing planes) is avoided.

The author demonstrated his balancing method using a numerical example. The results of this example are impressive. However, numerical examples are of limited value in evaluating the practical effectiveness of any rotor balancing procedure.

The author's balancing method is interesting and certainly has some advantages over both the modal and influence coefficient methods. However, it incorporates one of the principal disadvantages of modal balancing, in that predicted mode shapes and critical speeds are required (or, at least, no systematic method of empirical determination is discussed). Also, no attempt is made to deal with the difficulty of obtaining reasonable trial mass data at speeds above one or a few flexural critical speeds of a lightly damped system, since individual trial masses are apparently used for determining all of the influence coefficients. In comparison, the Unified Balancing Approach eliminates both of these disadvantages through the empirical

determination and use of modal trial mass sets. These particular considerations become especially important when considering the application of any flexible rotor balancing technique to practical rotor systems.

[105] Bigret, Roland, Curami, Andrea, Frigeri, Claudio, and Machhi, Nagelo, "Use of In-Field Computer for Balancing High Power Turbo-Machinery," ASME Paper No. 77-DET-11, (September) 1977.

The authors described a modified influence coefficient balancing method adapted for use with an in-field computer. The theoretical basis for this method is presented. As described, this method is essentially identical to the weighted least-squares method of Lund [55], with the added flexibility that the weighting matrix may be defined arbitrarily depending on the form of the residual vibration desired. For example, if it is of primary importance to reduce vibration at the running speed as compared to other data speeds, the weighting matrix can be defined so as to ensure that the residual vibration at the running speed is minimized with respect to the residual vibration at the other data speeds. An additional procedure is also mentioned whereby individual modal effects can be eliminated by the use of linear constraints and Lagrange multipliers. An analytical representation of this procedure is given. However, an explanation of the terms is omitted in lieu of a reference citation.

The in-field computer used relied on manual input of vibration data by means of a teletype. With this restriction, the use of an in-field computer provides an advantage in terms of convenience only, whereas the use of a computerized data acquisition system in conjunctioin with an in-field computer would provide the additional advantages of increased speed and accuracy.

A case study is described concerning the balancing of a 235-MW turbo-machine using the aforementioned procedure. This balancing

operation was apparently quite successful in terms of the residual vibration immediately adjacent to the bearings. The authors conclude that the use of the "speed weighting" in the balancing procedure was largely responsible for the success of the balancing test. The disadvantages of influence coefficient balancing are not as prominent when few modes are being balanced, as in this example.

The advantages of the Unified Balancing Approach over influence coefficient balancing in general (including the method described in this reference) have been discussed in considerable detail.

[106] Fujisawa, F., Siobato, K., Sato, K., Imai, T., and Shoyama, E., "Experimental Investigation of Multi-Span Rotor Balancing Using Least Squares Method," ASME Transactions, Journal of Mechanical Design, ASME Paper No. 79-WA/DE-1, 1980.

The authors described the results of an experimental investigation in which the least-squares influence coefficient method was used for balancing a multispan flexible rotor. They made the point that most of the literature concerned with flexible rotor balancing has placed the major emphasis on theoretical analysis rather than practical application. They claimed that most flexible rotor balancing methods, while effective for simple rotors, are not practical for more complex rotor systems. Although multi-span rotors are of considerable practical importance (e.g., modern steam turbine-generator sets), there is scant reference in the literature to balancing of multi-span flexible rotors.

The rotor system used for the experiments described in this paper consisted of five rotor spans supported by ten journal bearings (two bearings per span). For purposes of balancing, vibrations were measured in both the horizontal and vertical transverse directions at each bearing.

The authors presented a brief derivation of the least-squares balancing equations and a description of the balancing procedure, followed by discussions of the experimental apparatus and results. The experiments involved the use of unbalance in a single span, as well as unbalance in all spans. The authors reported good results in all cases. The characteristics of the bearings were found to be

sufficiently anisotropic to require the use of both horizontal and vertical vibration data to achieve a good balance.

Although the actual balancing approach described in this paper is not new, the application is interesting and of practical significance. The advantages of the Unified Balancing Approach over influence coefficient balancing methods have been discussed in considerable detail.

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## BIOGRAPHICAL SKETCH

Mark Sandor Darlow was born on July 31, 1949. He completed his primary and secondary education at the Dade County Public School System in Miami, Florida. In March 1972 he received the degree of Bachelor of Science with honors in mechanical engineering from the University of Florida. He immediately entered graduate school at the University of Florida, and, in August 1973, received the degree of Master of Engineering with a major in mechanical engineering. The title of his master's thesis was "Torsional Stability Analysis of a Helicopter Drive System."

Mr. Darlow then obtained a position of Dynamics Engineer with Mechanical Technology Incorporated (MTI) of Latham, New York. He subsequently attained the position of Project Engineer with MTI. While at MTI, Mr. Darlow worked in research and development and authored numerous publications in the areas of bearing defect analysis, rotordynamics, high-speed rotor balancing, acoustic emission, squeeze-film damper technology and dynamic testing of elastomers. As a project engineer, he carried project level responsibility, including the supervision of technicians and other engineers.

In January 1978, Mr. Darlow left MTI to return to the University of Florida to pursue the degree of Doctor of Philosophy in mechanical engineering. During this period, he has also held the

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Mr. Darlow currently resides in Gainesville with his wife, Myra, and son, Adam.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

George N. Sandor, Chairman
Professor of Mechanical Engineering

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Robert B. Gaither
Professor and Chairman of Mechanical Engineering

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George Piotrowski
Associate Professor of Mechanical Engineering

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